Bernoulli Numbers and Sums of Powers

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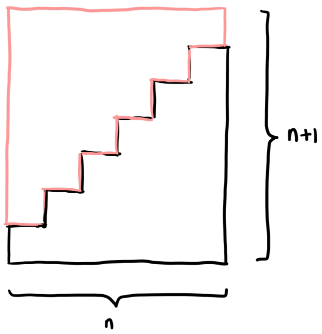
1 Sums of powers

Question 1.1. *What is* $1 + 2 + \ldots + 100$*?*

This question was posed to Gauss by his elementary school teacher, who wanted him to spend some time on it. Very quickly, Gauss realised that you can pair up the numbers 1 and 100, 2 and 99, and so on. There are 50 pair, and each pair sums to 101, so the value of the sum is $50 \times 101 = 5050$.



Generalising this arguement, we obtain the following.

$$1 + 2 + \ldots + n = \frac{1}{2}n(n+1)$$

Question 1.2. *What is* $S_p(n) := 1^p + 2^p + \ldots + n^p$*?*

Firstly, we observe that $(k+1)^p - k^p$ is a polynomial in k of degree $p-1$. We can use this to obtain a formula for $S_p(n)$ from $S_1(n), \ldots, S_{p-1}(n)$, which we demonstrate with an example.

Example 1.3. We claim that

$$S_2(n) = \frac{1}{6}n(n+1)(2n+1).$$

Indeed, we have that $(k+1)^3 - k^3 = 3k^2 + 3k + 1$. Thus,

$$\sum_{k=1}^{n}(k+1)^3 - k^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

The left hand side is a telescoping sum. Cancelling terms, we are left with

$$(n+1)^3 - 1 = 3S_2(n) + 3S_1(n) + n.$$

Substituting Gauss' formula for $S_1(n)$ and rearranging, we obtain the claimed formula.

In particular, we deduce (by induction), that $S_p(n)$ is a polynomial in n of degree $p+1$. It remains to find the coefficients.

Theorem 1.4 (Faulhaber's formula)**.** *For $p \geq 1$ and $n \geq 1$,*

$$S_p(n-1) = \frac{1}{p+1}\sum_{k=0}^{p}\binom{p+1}{k}B_k n^{p+1-k},$$

where B_k is defined recursively as follows: $B_0 = 1$ and for $n \geq 1$,

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0.$$

Example 1.5. The first few B_k can be calculated to be

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \dots$$

Remark 1.6. As a warning, the above formula is presented slightly differently depending on the source. The author presumes they are all equivalent, but it can be quite confusing. One should also note that there are two conventions for B_k, B_k^- and B_k^+, which differ at $k = 1$ by a sign. The formula differs if we use B_k^+. We will be using B_k^- throughout.

As part of a long series of theorems not named after the person who first wrote them down, or after the first person who proved it, Faulhaber's formula was written in this form by Daniel Bernoulli, a Swiss mathematician, and was published posthumous in *Ars Conjectandi*, 1713. In parallel, a similar formula was presented in Takakazu Seki's work *Katuyousanpou*, published in Japan in 1712. Seki had studied the methods of algebra developed in China, known as *Tianyuan Shu*. Unfortunately, much documentation of Chinese mathematics has been lost (Sad!), but for instance, Pascal's triangle and binomial coefficients were known by Yang Hui in the 11th century. Adapting and expanding these methods, Seki laid the foundations of *wasan* - traditional Japanese mathematics - which continued development until the importation of Western mathematics in the 19th century. Whether there was any connection between these discoveries is disputed, but in any case, it wasn't until 1834 that a rigorous proof was provided by Carl Jacobi.

Definition 1.7. The B_k are called the *Bernoulli numbers*, or very occasionally, the *Seki-Bernoulli numbers.*

2 Generating functions

We aim to prove Faulhaber's formula, for which we will need *generating functions.*

Define the infinite sum

$$f(t) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Remark 2.1. The $k!$ ensures that the coefficents don't grow too quickly, and hence that the series converges for all $t \in \mathbb{C}$. Here, we will ignore all convergence issues, but it should be noted that convergence issues are very important.

Proposition 2.2. *For $t \in \mathbb{C}$,*

$$f(t) = \frac{t}{e^t - 1}.$$

Remark 2.3. Sometimes you'll see this as the definition of B_k.

Proof. We have

$$e^t = \sum_{k=0}^{\infty}\frac{1}{k!}t^k.$$

So shifting indices,

$$e^t - 1 = \sum_{k=0}^{\infty}\frac{1}{(k+1)!}t^k.$$

Given two infinite series $\sum_{i=0}^{\infty} a_i$ and $\sum_{j=0}^{\infty} b_j$, we can multiply them to obtain

$$\sum_{i=0}^{\infty} a_i \sum_{j=0}^{\infty} b_j = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}.$$

Applying this to $e^t - 1$ and $f(t)$, we have

$$\begin{aligned}
(e^t - 1)f(t) &= \sum_{i=0}^{\infty} \frac{1}{(i+1)!} t^i \sum_{j=0}^{\infty} \frac{B_j}{j!} t^j \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{1}{(n-k+1)!} t^{n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\sum_{k=0}^{\infty} \binom{n+1}{k} B_k \right) t^{n+1} \\
&= t.
\end{aligned}$$

where the last line is given by the recursive definition of B_i. Rearrange for result. □

What can we do with this? Note that

$$\frac{d^n f}{dt^n} = \sum_{k=n}^{\infty} B_k k(k-1)\dots(k-n+1) \frac{t^k}{k!}.$$

Hence $\frac{d^n f}{dt^n}(0) = B_n$. For example, we have that

$$f'(t) = \frac{1}{e^t - 1} - \frac{te^t}{(e^t-1)^2} = \frac{e^t - 1 - te^t}{(e^t - 1)^2}.$$

As $t \to 0$, this tends to $-1/2$, as expected.

Remark 2.4. This is a very inefficient method for calculating Bernoulli numbers. David Harvey designed a fast algorithm using the Chinese remainder theorem.

Remark 2.5. Generating functions are ubiquitous in mathematics, and appear in many deep theorems, including statistics, complex analysis, and algebraic geometry.

3 Proof of Faulhaber's formula

We present a proof of Faulhaber's formula, following [**?**]. Since we have that

$$e^{mt} = \sum_{k=0}^{\infty} \frac{(mt)^k}{k!},$$

it follows

$$\sum_{m=0}^{n-1} e^{mt} = \sum_{m=0}^{n-1} \sum_{k=0}^{\infty} \frac{m^k t^k}{k!} = \sum_{k=0}^{\infty} \left(\sum_{m=1}^{n-1} m^k \right) \frac{t^k}{k!}.$$

But the left hand side is also a geometric series, so

$$\sum_{m=0}^{n-1} e^{mt} = \frac{e^{nt}-1}{e^t-1} = \frac{e^{nt}-1}{t}\frac{t}{e^t-1}$$
$$= \frac{e^{nt}-1}{t}\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

We have

$$\frac{e^{nt}-1}{t} = \frac{1}{t}\sum_{k=0}^{\infty}\frac{(nt)^{k+1}}{(k+1)!},$$

and so

$$\sum_{m=0}^{n-1} e^{mt} = \left(\sum_{k=0}^{\infty}\frac{n^{k+1}}{k+1}\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty} B_k\frac{t^k}{k!}\right)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\frac{1}{k-i+1}\binom{k}{i}B_i n^{k+1-i}\right)\frac{t^k}{k!}.$$

Equating coefficients,

$$S_p(n-1) = \sum_{i=0}^{k}\frac{1}{k-i+1}\binom{k}{i}B_i n^{k+1-i} = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+i}{i}B_i n^{k+1-i},$$

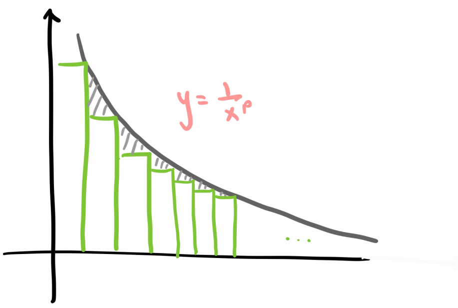
as required.

4 Further applications

The Bernoulli numbers appear in many surprising places. Wikipedia has a good summary. We will finish with one application, which also gives an alternative proof of Faulhaber's formula.

Question 4.1. *Does $\sum_{n=0}^{\infty}\frac{1}{n}$ converge? What about $\sum_{n=0}^{\infty}\frac{1}{n^2}$?*

It is known that $\sum_{n=0}^{\infty}\frac{1}{n^p}$ does not converge for $p=1$ and converges for $p>1$. An informal argument is to consider the integral $\int_{x=1}^{\infty} 1/x^p$. If $p=1$, this is $[\log x]_{x=1}^{\infty}$, which is undefined, while for $p>1$, this is $[x^{1-p}/(1-p)]_0^{\infty}$, which is defined. We conclude that the sum converges for $p>1$ and diverges for $p=1$.



This is not quite right since the integral and sum are not the same. In this case, $1/x^p$ is strictly decreasing and such an argument can be made into a proof, but this may not always be the case. Approximating sums with integrals is a key part of analytic number theory, and it is good to have good bounds on the error. One such formula for this is the *Euler-Maclaurin* formula.

Theorem 4.2 (Euler-Maclaurin formula)**.** *Say f is N times differentiable, $m, n \in \mathbb{N}$, $m < n$. Then*

$$\sum_{k=m}^{n} f(k) = \int_m^n f(x)dx + \frac{1}{2}(f(n) + f(m)) + \sum_{k=2}^{N} \frac{B_k}{k!}(f^{(k-1)}(b) - f^{(k-1)}(a)) + R(f, N),$$

where

$$R(f, N) = (-1)^{n+1} \int_m^n f^{(N)}(x) \frac{P_N(x)}{N!} dx,$$

and $P_N(x)$ are the periodic Bernoulli polynomials*.*

For now, we are not too concerned with the form of the remainder term (although this is important). Note that if f is a polynomial of degree $N-1$, then the remainder term vanishes, since the Nth derivative vanishes.

Sketch proof. The first step is to define the *Bernoulli polynomials*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k.$$

The key property is that $B_n(0) = B_n(1) = B_n$ for $n \neq 1$. The periodic Bernoulli functions P_n are defined to be the same as B_n on $[0, 1]$, and periodic with period 1.

The rest is an induction argument using integration by parts. Consider the integral $\int_k^{k+1}$ and a change of variables $v = P_1(x)$. Integrating by parts, we obtain

$$\int_k^{k+1} f(x)dx = B_1(1)f(k+1) - B_1(0)f(k) - \int_k^{k+1} f'(x)P_1(x)dx$$

Rearranging, we obtain the case $N = 1$. We continue, at each step changing variable by $v = \frac{1}{k!}P_k(x)$. □

Remark 4.3. This gives an alternative proof of Faulhaber's formula. Choose $f(x) = x^p$. Note that for $N = p + 1$, the remainder term vanishes. The result follows.