Tropical Brill-Noether Theory

- Chip Firing & Algebraic Curves (AMS)
- Sam Payne, teaching Tropical Brill-Noether Theory

1. Algebraic Curves

Curves admits maps to proj. space
Study of space of such maps $\rightsquigarrow$ Brill Noether Theory

rational maps

Important invariants of map $\varphi: C \longrightarrow \mathbb{P}^r$

- **Degree** - $\# \varphi^{-1}(H)$ H general hyperplane
- **Rank** - $\dim(\operatorname{span}(\operatorname{im}(\varphi)))$

- deg 2
- rank 2

Q: Does C possess a map deg d & rank r.
If so, how many?

Divisors

$\mathcal{O}_{X,x}$ UFD $\forall x \in X$

Recall: (Weil) divisors of (insert adjectives) scheme is
$\sum n_d [D]$
integral closed subscheme codim 1

On curves divisors are $\sum_{p \in C} D(p) \cdot p$
↳ smooth projective irred. ?

degree $\deg D = \sum_{p \in C} D(p)$

effective $D(p) \geq 0 \quad \forall p$

$D_1 \sim D_2$ if $D_1 - D_2 = \operatorname{div}(f) = \sum v_p(f)[p]$
equivalently for effective divisors $\exists\ \varphi: C \longrightarrow \mathbb{P}^n$ s.t. $D_i = \varphi^{-1}(H_i)$ $i = 1,2$
H_i hyperplanes.

Set of hyperplanes in $\mathbb{P}^r$ = r dim space
each hyperplane corresponds to an efficitive divisor in one linear equiv class.
$\operatorname{rank}(D) = \dim\{\text{set of effective divisors} \sim D\}$ $=: \ell(D)$

complete linear series

$D_i = D + \operatorname{div}(f_i)$ $i = 1,2$
$D_1 + D_2 = D + \operatorname{div}(f_1 + f_2)$

Fact: divisor D has rank r if for every effective divisor E of degree D, $D - E$ is equivalent to an effective divisor.

If D not equiv. to effective divisor, say it has rank -1.

Set of divisor classes is **Picard group** Pic^d

Brill Noether Varieties $W^r_d(C) := \{[D] \in \operatorname{Pic}^d(C) \mid \operatorname{rank}(D) \geq r\}$

Q: For which values of r and d is $W^r_d(C)$ non-empty
If so, how 'big' is it.

Riemann-Roch $\quad \operatorname{rank} D - \operatorname{rank}(K_C - D) = \deg D - g + 1$

canonical divisor

2. Chip Firing Game

Game played on a graph

Graphs - finite, connected, loopless but possibly w/ multi-edges

Def. Divisor D on graph G is formal $\mathbb{Z}$-linear combination of vertices of G, $D = \sum_{v \in V(G)} D(v) \cdot v$

↑

chip firing configuration

vertices with negative coeffs "in debt"

effective divisor

degree

Def. Chip-firing move

takes a divisor D to D', where

$$D'(w) = \begin{cases} D(v) - \mathrm{val}(v) & \text{if } w = v \\ D(w) + \#\text{ of edges b/w } w \,\&\, v & \text{if } w \neq v \end{cases}$$

Two divisors D, D' are Linearly equivalent if D' can be obtained from D by a sequence of chip firing moves.

Picard gp $|\mathrm{Pic}^d(G)| < \infty$

Kirchoff's matrix Tree Theorem

← more subtle

Rank D has rank at least r if $D - E \sim$ effective divisor $\forall E$ effective rank r

Chip firing game

→ opponent steals r chips

→ elliminate debt

Q: Does G possess a divisor of deg d & rank r ?
If so how many?

genus of graph first betti number

canonical divisor $K_G = \sum_{v \in V(G)} (\mathrm{val}(v) - 2) \cdot v$

Thm. (Riemann Roch for graphs) (Baker, Norine 07) G graph genus g, D diviser on G

$$\mathrm{rank}(D) - \mathrm{rank}(K_G - D) = \deg D - g + 1$$

neither implies other

but indicative of connection

3. Connecting the two worlds

Given a curve, want to torture it into a simpler object, retaining useful information

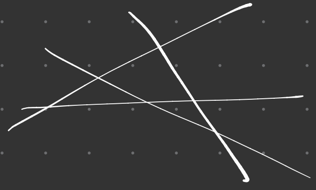
Extremely sketchy

Example family of curves : for $F(x,y,z)$ homog. poly deg 4

$$C_t = \{ (x,y,z) \in \mathbb{P}^2 \mid t \cdot F(x,y,z) + xyz(x+y+z) = 0 \}$$

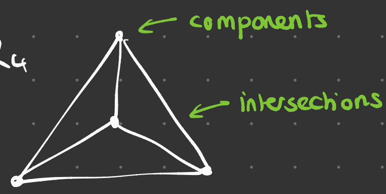
Almost all values of t, $C_t \subset \mathbb{P}^2$ is a smooth plane quartic

$t = 0$



dual graph $\rightsquigarrow$

K_4



In order for G to be used to state properties of G, family must satisfy some technical hypothesis
↳ think of C_t as a single curve over the DV field $K((t))$
DVR $K[[t]]$

→ A curve $\mathscr{C}$ over R is a strongly semistable model for C if $\mathscr{C}$ regular, proper & flat over R, general fibre is C, special fibre C_0 reduced & has only ordinary double points as singularities

→ Properties of C hold for C_t t in some dense open set

Consider 1 parameter family of divisors
With sufficiently strong hypothesis, they restrict to smooth pr of C_0
$\rightsquigarrow$ well defined map divisors of $C(K) \longrightarrow V(G)$
$\rightsquigarrow$ preserves linear equivalence → why?

Upshot: have map $\mathrm{Trop}: \mathrm{Pic}_K(C) \longrightarrow \mathrm{Pic}(G)$
↑ coeffs in discretely valued field $\mathbb{C}((t))$

Thm. (Baker's Specialization Lemma, '08) Let D be a K divisor on C
Then

$$\mathrm{rank}(D) \leq \mathrm{rank}(\mathrm{Trop}\, D)$$

Link to Tropical curves $\rightsquigarrow$ some divisors of particular degree & rank don't exist
solution is to refine graph
→ in limit get metric graph

Back to Q: For which values of r and d is $W^r_d(C)$ non-empty
If so, how 'big' is it.

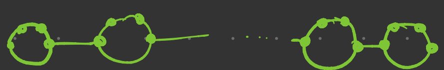
Thm. (Brill-Noether Theorem) Let C be a general curve of genus g, define

$$\rho(g,r,d) = g - (r+1)(g-d+r)$$

1. If $\rho(g,r,d) < 0$ $\quad W^r_d(C)$ is empty
2. If $\rho(g,r,d) \geq 0$ $\quad \dim W^r_d(C) = \rho(g,r,d)$
   $\leq$ in general

Alternate pf by chip firing given by Cools, Draisma, Payne, Robeva 2012

Idea: by Baker's specialization lemma, suffices to exhibit a family of graphs genus g, closed under refinement, that possess no divisor of deg d, rank at least r

Chain of loops 

$\xrightarrow{\text{more work}}$ also get second part