Cohomology and Vector Bundles

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Given topological spaces X, Y, we would like to know whether they are homeomorphic to one another. It turns out this is hard, so instead we consider various invariants, which are preserved under homeomorphisms. For X a suitable space, we can define *cohomology groups*, $H^k(X)$ for $k \in \mathbb{Z}_{\geq 0}$. In a very imprecise sense, cohomology captures the k-dimensional "holes" in a space.

Much of this talk is lifted from lectures on differential geometry by Jack Smith and algebraic topology by Ivan Smith.

1 Tangent spaces and vector bundles

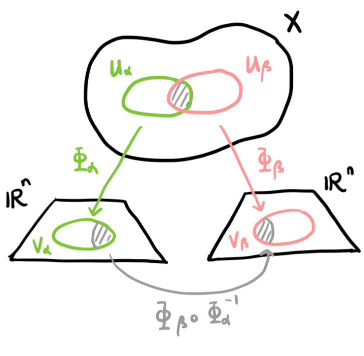
This is a seminar on K-theory, so we should learn about vector bundles. On the way, we will pick up de Rham cohomology, and use this to motivate some more general notions.

Definition 1.1. A topological space X is an *n-manifold* if

1. for each point $p \in X$, there is an open neighbourhood U of p, an open set $V \subseteq \mathbb{R}^n$, and a homeomorphism $\varphi : U \to V$.
2. X is Hausdorff and second countable.

We call φ an *chart* around p. If we have φ_α, φ_β overlapping charts, we have *transition functions*

$$\varphi_\beta \cdot \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$



Definition 1.2. A *smooth n-manifold* is an n-manifold with smooth transition functions.

Heuristically, what this means is that locally on an n-manifold X, X "looks like" $\mathbb{R}^n$. Any function from X to a space Y can be considered locally as a function from an open set of $\mathbb{R}^n$ to Y. Hence notions such as the differentiability of a real valued function makes sense. The reader is encouraged to think about what the definition of a smooth map between smooth manifolds should be.

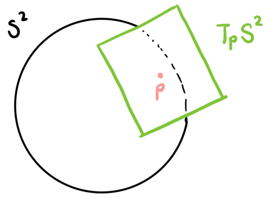
Definition 1.3. Smooth manifolds X and Y are *diffeomorphic* if there are smooth maps $f : X \to Y$ and $g : Y \to X$ with $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$.

Example 1.4. The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is a smooth n-manifold. Charts are given by stereographic projection.

Note that at each point x of S^n, we have a *tangent space*, T_xS^n, given by the n-dimensional hyperplane that is tangent to S^n at x, translated to the origin[1]. In other words,

$$T_xS^n = \{y \in \mathbb{R}^n | \langle x, y\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$.



For an *embedded* manifold in $X \subseteq \mathbb{R}^n$, we can consider the tangent space at $x \in X$ to be given by the hyperplane which is the "first order approximation" to X at x (translated to the origin). For abstract smooth manifolds, we should have an intrinsic definition of the tangent space. We will not be too concerned with the definitions, but it is here for those who want it.

Definition 1.5. A *curve based at* p is a smooth map of the form $\gamma : I \to X$ which sends 0 to p, where I is an open neighbourhood of 0.

Two curves γ_1, γ_2 based at p are said to *agree to first order* if there exists charts φ about p such that

$$\frac{\partial}{\partial t}(\varphi \circ \gamma_1)(0) = \frac{\partial}{\partial t}(\varphi \circ \gamma_2)(0).$$

Definition 1.6. The *tangent space* to X at p is

$$T_pX = \{\text{curves based at } p\}/\text{agreement to first order}.$$

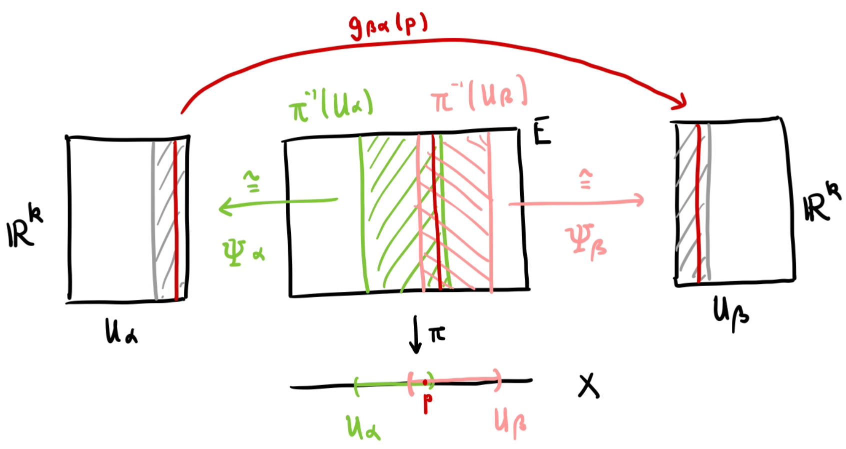
Elements of T_pX are called *tangent vectors* at p.

One can see (and perhaps try to prove[2]) that T_pX is isomorphic to $\mathbb{R}^n$ as a real vector space. However, T_pX has more information than just a collection of vector spaces (which is very boring on its own). In particular, we should be able to capture how these vector spaces are "glued together". We naturally arrive at the notion of a *vector bundle*.

Definition 1.7. Let X be a topological space. A *vector bundle* of rank d, is the following data:

1. A manifold E, called the *total space*
2. A smooth map $\pi : E \to B$, called the *projection*.
3. An open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of X.
4. For each α, a diffeomorphism
$$\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k,$$
called *local trivialisations*, such that
    - $\mathrm{pr}_1 \circ \Psi_\alpha = \pi$ on $\pi^{-1}(U_\alpha)$.
    - For all α, β, the map $\Psi_\beta \circ \Psi_\alpha^{-1}$ on $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$ has the form $(b, v) \mapsto (b, g_{\beta\alpha}(b)v$ for some map $g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$.

The $g_{\beta\alpha}$ are called *trivialisation functions*. The fibres $\pi^{-1}(p)$ are denoted E_p.



This is a mouthful of a definition, but it is summarised by a picture

Remark 1.8. We may wish to consider vector bundles on manifolds which are not smooth. In this case, we have that π is merely continuous and the trivialisations are homeomorphisms.

Remark 1.9. You can replace $\mathbb{R}$ with $\mathbb{C}$ to obtain a *complex* vector bundle.

Example 1.10 (The trivial line bundle and the Mobius bundle over S^1)**.** S^1 can be covered by two open sets isomorphic to $(0,1)$ with two intersections. Consider line bundles (vector bundles rank 1) trivialised over these open sets. The transition function over one intersection set can (without loss of generality) be set to $g_{01} = 1$. For the other trivilisation, we have two options: $g_{10} = 1$ or $g_{01} = -1$. This gives the *trivial line bundle* and the *mobius bundle* respectively.

Example 1.11 (Tautological bundle over Grassmannian)**.** The *Grassmannian* $G(k,n)$ parameterises k-dimensional hypersurfaces in $\mathbb{R}^n$. For $k = 1$, we obtain the projective space $P\mathbb{R}^{n-1}$ We have a *tautological bundle*

$$\tau = \{(x,h) \in \mathbb{R}^n \times G(k,n) : x \in h\} \to G(k,n),$$

where the map is given by projection. The fibre at every point p is the k-dimension hyperplane corresponding to p. We leave it to the exercise to the reader that this is naturally a vector bundle (i.e. local trivialisations exist). Analogously, you can replace $\mathbb{R}$ by $\mathbb{C}$.

Suppose $\pi : E \to X$ and $\pi' : E' \to X$ are vector bundles over X. Then there is a vector bundle whose fibres at each point $p \in X$ is given by $E_p \oplus E'_p$. We call this the *Whitney sum* $E \oplus E'$ of E and E'. This is an important definition for later. We leave it as an exercise to show the definition makes sense.

Similarly, we obtain the *tensor product* $E \otimes E'$ and *dual vector bundle* E^* by tensoring or taking the dual vector space[3] at each fibre respectively.

Definition 1.12. The r-th *wedge/exterior power* of a vector space V, denoted $\wedge^r V$, is spanned by

$$\{v_1 \wedge \ldots \wedge v_r | v_i \in V\}$$

with the relation

$$v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)} = \mathrm{sign}(\sigma) v_1 \wedge \ldots \wedge v_r.$$

[1]We often draw pictures without translating to the origin, but this is necessary to have the tangent space be a vector space.

[2]Read: the author did not want to write out the proof.

[3]$E_p^* = \mathrm{Hom}_{\mathbb{R}}(E_p, \mathbb{R})$.

You should think of an element of $\wedge^r V$ as a parallelopiped with vectors v_i in the corners. Swapping the vectors gives you "negative volume". Taking the r-th exterior power fibrewise gives a vector bundle $\wedge^r E$.

2 De Rham cohomology

Fix smooth manifold X.

Definition 2.1. The *cotangent bundle* of X, denoted T^*X, is the dual to the tangent bundle.

Let's describe this more concretely. For $p \in X$, consider an open neighbourhood $U \subseteq X$, and smooth function $f : U \to \mathbb{R}$. If X has local coordinates x_i around p, then there is a natural basis of T_pX given by $(\partial x_i)_i$[4]. We obtain an element of T_p^*X defined by

$$f \mapsto \left(\sum a_i \partial_{x_i} \mapsto \sum a_i \frac{\partial f}{\partial x_i}\right).$$

We call this element $d_p f$.

Remark 2.2. Given our intrinsic definition of the tangent space, we can also write this as

$$[\gamma] \mapsto \frac{\partial}{\partial t}(f \circ \gamma)(0).$$

Patching the $d_p f$ together, we obtain a *section* of T_p^*X, i.e. a smooth map $s : X \to T_p^*X$ such that $\pi \circ s = \mathrm{id}_X$. We call sections of T_p^*X *one forms*. If $f : X \to \mathbb{R}$ is defined globally, we obtain a one form df.

Definition 2.3. An *r-form* of X is a section of $\wedge^r T^*X$. The space of r-forms on X is denoted $\Omega^r(X)$.

Remark 2.4. One can think of one forms as infinitesimal arrows, two forms as infinitesimal areas, three forms as infinitesimal volumes, and so on.

As a notational point, denote by dx_i the dual to ∂_{x_i}. We write $\sum_I \alpha_I dx_{i_1} \wedge dx_{i_r} = \alpha_I dx^I$, where the sum runs over subsets $I = \{i_1, \ldots, i_r\} \subseteq [n] = \{1, \ldots, n\}$.[5]

Definition 2.5. The *exterior derivative* of a r-form $\alpha = \alpha_I dx^I$ is

$$d\alpha = d\alpha_I \wedge dx^I.$$

One should check that this is well-defined (independent of coordinates).

Proposition 2.6. $d^2 = 0$.

Proof. This is from the symmetry of partial differentials. Precisely, for $\alpha = \alpha_I dx^I$, we have

$$d^2\alpha = d\left(\frac{\partial \alpha_I}{\partial x^j} dx^j \wedge dx^I\right) = \frac{\partial^2 \alpha_I}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^I.$$

Since $\frac{\partial^2 \alpha_I}{\partial x^k \partial x^j}$ is symmetric in j, k but $dx^k \wedge dx^j$ is antisymmetric. □

Definition 2.7. A *cochain complex* is a sequence $\{C^r\}_{r \in \mathbb{Z}_\geq}$ of abelian groups and homomorphisms $d^r : C^r \to C^{r+1}$ such that $d^{r+1} \circ d^r = 0$. We often suppress the r in d^r.

Definition 2.8. A form α is *closed* if $d\alpha = 0$ and *exact* if there exists a form β such that $d\beta = 0$. We denote the space or closed and exact r-forms $Z^r(X)$ and $B^r(X)$ respectively.

[4] ∂_{x_i} is the (unique) tangent vector mapping to the standard basis vector e_i under the map $T_pX \to \mathbb{R}$, $\gamma \mapsto \frac{\partial}{\partial t}(\varphi \circ \gamma)(0)$

[5] Yes, the sum sign has disappeared. The sum is implicit in the notation.

Definition 2.9. Note that $d^2 = 0$ implies $B^r(X) \subseteq Z^r(X)$. A cochain complex is *exact* if $B^r(X) = Z^r(X)$.

Definition 2.10. The r-th de-Rham cohomology group of X is

$$H^r_{\mathrm{dr}} = Z^r(X)/B^r(X).$$

Why suffer through this? Consider S^1. We claim that

$$H^r_{dR}(S^1) = \begin{cases} \mathbb{R} & r = 0, 1 \\ 0 & \text{otherwise.} \end{cases}$$

Parametrise S^1 by angle θ. Heuristically, we have a closed one-form $d\theta$, which is not exact - if $d\theta = df$, then $f = \theta + a$, but this is not continuous. So we see that $H^1(S^1)$ has captured the "hole" in S^1.

We briefly outline the proof. Any 1-form on S^1 can be written uniquely on $f(\theta)d\theta$ and all 1-forms are closed. This is because S^1 is one-dimensional, but one should prove this carefully. Define a map $I : \Omega(S^1) \to \mathbb{R}$ and $f(\theta)d\theta \mapsto \int_0^{2\pi} f(\theta)d\theta$. This is linear and non-zero, hence surjective. It remains to show that $\ker(I) = B^1(S^1)$.

In order for this to be useful, we should have invariance under homeomorphism. It turns out de Rham cohomology is invariant under something weaker, namely *homotopy*[6]. This means it is not as refined an invariant as it could be (since there exist spaces which are homotopic but not homeomorphic, such as the circle and the annulus), but this is a sacrifice we are willing to make.

Remark 2.11. One could ask if two spaces with the same cohomology groups are homotopic. This is not true.

Definition 2.12. Given a smooth map $f : X \to Y$, the *derivative* of f at $p \in X$ is the map $D_pX : T_pX \to T_{f(p)}Y$ defined by $[\gamma] \mapsto [f \circ \gamma]$.

Definition 2.13. Given a smooth map $f : X \to Y$, the map $(D_pf)^\vee : T^*_{f(p)}Y \to T^*_pX$, dual to D_p, is the *pullback* by f, denoted f^*. This induces a map $f^* : \Omega^r(Y) \to \Omega^r(X)$.

Proposition 2.14. *For $f : X \to Y$ smooth, α an r-form on Y, $d(f^*\alpha) = f^*(\alpha)$. Hence f^* descends to a map $f^* : H^r_{dR}(Y) \to H^r_{dR}(X)$.*

It is clear that $(\mathrm{id})^* = \mathrm{id}$ and that $(f \circ g)^* = g^* \circ f^*$.

Theorem 2.15. *If f, g are homotopic, then $f^* = g^*$.*

This is well beyond the scope of this talk, but it follows fairly easily from *Cartan's magic formula*. We leave it as an exercise to show homotopy invariance from here (this is not hard).

Remark 2.16 (For experts). De Rham cohomology has many advantages. It can be computed for non-trivial spaces, and the generators of the cohomology groups are explicit. The ring structure comes directly from the wedge product. Poincare duality follows from Stoke's theorem and Liebnitz rule. This is to be expected, since one has the full-force of calculus available to them, but the payoff is that we are restricted to smooth manifolds. Two circles glued together at a point is not a manifold, let alone a smooth one (exercise: why?), but an algebraic topologist would definitely like to work with such a space. Similarly, an algebraic geometer would like to work with singular varieties, and one has to reconstruct the theory to deal with these.

3 Cohomology in general

There are many ways to define cohomology. One can recall, or look up, or take on faith, examples such as simplicial or singular cohomology[7]. It is not at all clear that they should give the same thing, and

[6]Two maps $f, g : X \to Y$ are *homotopic* if there exists continuous $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Two spaces X, Y are homotopic if there exists $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity.

[7]A good source for this is Hatcher's Algebraic Topology.

the connections are deep[8]. This zoo of cohomologies should indicate that there is some kind of unifying theory, which brings us to the Eilenberg–Steenrod axioms. We will not prove anything, and choose to simply believe that a good cohomology theory should have these properties.

Definition 3.1. A *pair of spaces* is a pair (X, A) with $A \subseteq X$. A *map of pairs of spaces* $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ with $f(A) \subseteq B$.

A *generalised cohomology theory* is an assignment

$$(X, A) \to h^*(X, A) = \bigoplus_{i \in \mathbb{Z}} h_i(X, A)$$

satisfying the following

1. (Functoriality.) A map $f : (X, A) \to (Y, B)$ induces $f^* : h^*(X, A) \to h^*(Y, B)$ such that $(\mathrm{id})^* = \mathrm{id}$ and that $(f \circ g)^* = g^* \circ f^*$.
2. (Homotopy invariance.) If f, g are homotopic as maps of pairs, then $f^* = g^*$.
3. (LES of pairs) Write $h^i(X) = h^i(X, \varnothing)$. Then there exist maps $h^i(A) \to h^{i+1}(X, A)$ such that
$$\ldots \to h^i(X, A) \to h^i(X) \to h^i(A) \to h^{i+1}(X, A) \to \ldots$$
is exact, where the other maps are induced by the natural inclusions.
4. (Excision.) If $\bar{Z} \subseteq \mathrm{int}(A)$, then the inclusion induces isomorphism
$$h^*(X \setminus Z, A \setminus Z) \to h^*(X, A).$$
5. (Unions.) The natural inclusions induces isomrophism
$$\bigoplus_{\alpha} h^*(X_\alpha) \to h^*(\coprod_{\alpha} X_\alpha).$$

The groups $h^*(\mathrm{pt})$ are called the *coefficients* of the theory. If the coefficents are not specified, we are working with $\mathbb{Z}$ coefficients.

Theorem 3.2. *If h^*, k^* are generalised cohomology theories, then for a suitably nice[9] pair of spaces (X, A), if $\Phi : h^* \to k^*$ is a natural transformation giving $h^*(\mathrm{pt}) \cong k^*(\mathrm{pt})$, then we have $h^*(X, A) \cong k^*(X, A)$.*

Remark 3.3. We can reverse all the arrows, and obtain a *homology* theory. In many cases (such as simplical or singular), we start with homology.

This is a lot of definitions, so I will finish with an example.

Proposition 3.4. *The sphere S^2 is not homotopic (hence not homeomorphic) to the torus T^2.*

Proof. We have

$$H^*(S^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$H^*(T^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ \mathbb{Z}^2 & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

□

Remark 3.5. The best way to do the above calculations is the Mayer-Vietoris sequence but an interesting exercise might be to calculate this from the axioms.

[8]A nice proof of the connection between Cech cohomology and de Rham cohomology using (very tame examples of) spectral sequences is given in Differential Forms and Algebraic Topology.

[9]Cell complex and subcomplex, for example.