Enumerative Geometry

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1 The problem of Apollonius

Question 1.1. *Consider three circles in a plane. How many circles are tangent to all three circles?*

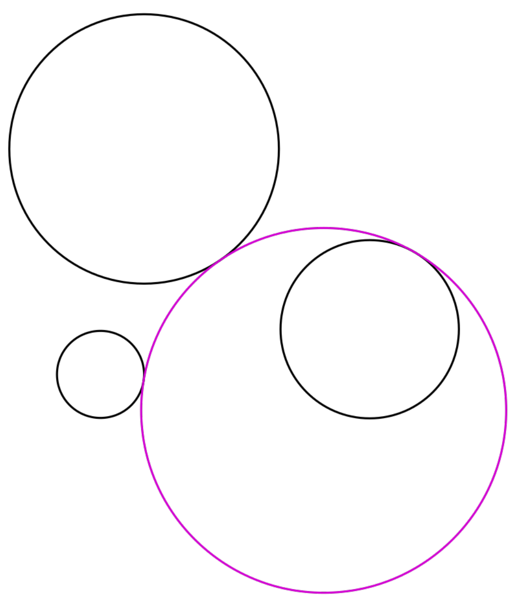

Figure 1. One solution (in purple) to Apollonius problem

There are many approaches to this problem, first posed by Apollonius of Perga (c. 262 BC – c. 190 BC). We will make use of coordinate geometry in $\mathbb{R}^2$.

Let the centre of the circles be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and radii be r_1, r_2, r_3 respectively. We look for a circle with centre (x_s, y_s) with radius r_s. Two circles are tangent if and only if the distance between the two centres is equal to the sum or difference of the two radii (depending on whether the circles are externally or internally tangent). This gives us a system of equations for x_s, y_s, r_s.

$$(x_1 - x_s)^2 + (y_1 - y_s)^2 = (r_1 \pm r_s)^2 \tag{1}$$
$$(x_2 - x_s)^2 + (y_2 - y_s)^2 = (r_2 \pm r_s)^2 \tag{2}$$
$$(x_3 - x_s)^2 + (y_3 - y_s)^2 = (r_3 \pm r_s)^2 \tag{3}$$

Multiplying out Eq. (1) - Eq. (3), we have $x_s^2 + y_s^2$ appearing on the left hand side and r_s^2 on the right hand side of all three equations. Subtracting Eq. (1) from Eq. (2) and Eq. (3), we obtain expressions linear in x_s, y_s, r_s.
For example, the first two equations give

$$x_1^2 + y_1^2 - x_2^2 - y_2^2 - 2(x_1 - x_2)x_s - 2(y_1 - y_2)y_s = r_1^2 - r_2^2 + 2(r_1 - r_2)r_s \tag{4}$$

and similarly using Eq. (1) and Eq. (3) we obtain a linear equation in x_s, y_x, r_s[1]. Assuming these equations are non-degenerate, we may rearrange these linear equations to obtain

$$x_s = M + Nr_s \tag{5}$$
$$y_s = P + Qr_s \tag{6}$$

where M,N,P,Q are known rational functions of x_i, y_i, r_i[2].
We may then substitute this into Eq. (1). This gives a quadratic in r_s which has at most two solutions, except in degenerate cases. We have eight ways of choosing the signs in Eq. (1) - Eq. (3), thus we may think we have (in general) at most 16 solutions.
However, note that if (r_s, x_s, y_s) is a solution, so is $(-r_s, x_s, y_s)$ but with different choices of signs. Thus we have at most 8 solutions, except in degenerate cases where we have infinitely many solutions. Finding the conditions

[1] These equations are not linear in x_1 etc. but these variables are known, so this is not a problem.

[2] A *rational function* is a function of the form p/q where p, q are polynomials

for the configuration to be degenerate is left as an exercise for the reader, but one may show that they are somehow 'rare'. We lose solutions when we have, for example, double roots or imaginary solutions.

Remark. This result is both satisfying and unsatisfying. *In general* we have *at most* 8 solutions. It would be even nicer if we always had exactly 8 solutions, but unfortunately this is not the case for circles in the plane.

Note that the problem would partially be fixed if we allowed *complex* solutions, since in $\mathbb{C}$, *most* degree d polynomials have exactly d roots. A circle with imaginary radius is much harder to visualise, but the payoff is bountiful. Thus from this point on, we will always work with $\mathbb{C}$ instead of $\mathbb{R}$.[3]

Remark. Here we have taken a geometric question and reduced it to counting the number of intersections of some curves, given by Eq. (1) - Eq. (3). This is a general technique, and studying intersections of curves is a rich area of mathematics.

2 Bezout's Theorem

We have seen that studying intersections of curves is useful, and in this section we will consider plane curves in particular.

Example 2.1. Two distinct lines in the plane (degree 1 curves) intersect at one point, unless they are parallel.

Example 2.2. We define a *quadratic curve* in the plane $\mathbb{C}^2$ to be the set defined by $ax^2 + by^2 + cxy + dx + ey + f = 0$ with at least one of $a, b, c \neq 0$ (degree 2 curve). We intersect a line with a quadratic. By applying a rotation and translation, without loss of generality the line is $y = 0$. Thus we are counting the solutions to $\alpha x^2 + \beta x + \gamma$ with $\alpha \neq 0$. This has two solutions, except when we have a double root.

The observant reader may realise that $1 \times 1 = 1$ and $2 \times 1 = 2$, from which we may conjecture that the intersections of degree n and degree m curve meet at nm points. This is not true, even the simplest case of two distinct

[3]More generally, we may work with an *algebraically closed* field of *characteristic* 0. Asking what happens when you remove these conditions is the avenue to a fabulous leap from classical to modern algebraic geometry, but since this footnote space is too small, this is a story for another time.

lines. However, we can 'fix' this result by saying that parallel lines meet 'at infinity'. We will now define this notion more formally.

Definition 2.3. The *projective space* of dimension n is

$$\mathbb{PC}^n = \mathbb{P}^n = \{\text{lines in } \mathbb{C}^{n+1} \text{ through the origin}\}$$

Example 2.4. Analogously, we can define $\mathbb{PR}^n$. We can define a line in $\mathbb{R}^2$ by any non-zero point, or by any point on the unit circle. However two antipodal points on the circle give the same line. So $\mathbb{PR}^1$ is the circle with opposite points identified. It turns out that $\mathbb{PR}^1$ is equivalent to the circle, but this is not a general result! In particular, $\mathbb{PR}^2 \ncong$ sphere.

We may specify any line in $\mathbb{C}^{n+1}$ using a non-zero point, say $(a_0, ..., a_n)$. We denote this line (which is a *point* in $\mathbb{PC}^n$) by $[a_0 : ... : a_n]$.

Here we give a big WARNING. These 'coordinates'[4] for $\mathbb{PC}^n$ are *ambiguous*! Given any non-zero constant $\lambda \in \mathbb{C}$, $(a_0, ..., a_n)$ and $(\lambda a_0, ..., \lambda a_n)$ define the *same line*. So $[a_0 : ... : a_n]$ and $[\lambda a_0 : ... : \lambda a_n]$ specify the *same point* in $\mathbb{PC}^n$.

Remark. We should convince ourselves that this is the extension to $\mathbb{C}^n$ that we want. Note that in $\mathbb{CP}^n$ we have a set

$$U = \{[a_0 : ... : a_n] \in \mathbb{CP}^n | a_0 \neq 0\} \subset \mathbb{CP}^n$$

We should check that this set is well-defined. Note that if $a_0 \neq 0$, then for all $0 \neq \lambda \in \mathbb{C}, \lambda a_0 \neq 0$. So the property $a_0 \neq 0$ holds for any representation of $[a_0 : ... : a_n]$, as required.

Now for any such $[a_0 : ... : a_n]$, we may divide each entry by $a_0 \neq 0$ to obtain $[1 : a'_1 : ... : a'_n]$. Thus points in U correspond to points in

$$\mathbb{C}^n = \{(a'_1 : ... : a'_n) | a'_1, ..., a'_n \in \mathbb{C}\}$$

As an exercise, the reader may want to convince themselves that $\mathbb{CP}^n$ is really an extension of $U \cong \mathbb{C}^n$ in this way, and that the set $\mathbb{CP}^n - U$ is 'small'.

We now work in the projective plane $\mathbb{P}^2 = \mathbb{PC}^2$, coordinates $[T : X : Y]$. We want to consider sets defined by equations $f(x) = 0$, where f is a polynomial, but we immediately have a problem. The ambiguity of our coordinates mean evaluating polynomials do not make sense. If we try to evaluate $X + 2$ at $P = [1 : 1 : 2]$, we have $1 + 2 = 3$. But $P = [2 : 2 : 4]$, and $2 + 2 = 4$. So we will work with a particular subset of polynomials.

[4] called *homogeneous coordinates*.

Definition 2.5. An *homogeneous polynomial* of degree n in variables X, Y, T is a sum of terms of degree n, i.e. $cT^aX^bY^c$ such that $a + b + c = n$.

Lemma 2.6. *If f is a homogeneous polynomial of degree n, then*

$$f(\lambda t, \lambda x, \lambda y) = \lambda^n f(t, x, y)$$

Proof. Exercise. □

As a consequence, for $P = [t : x : y]$ and f a homogeneous polynomial of degree n, $f(P)$ does *not* make sense, but we *can* define the set

$$V(f) = \{[t : x : y] : f(t, x, y) = 0\}$$

Definition 2.7. A *degree n curve* in $\mathbb{P}^2$ is a set of the form $V(f)$, where f is a homogeneous polynomial of degree n.

Example 2.8. We define a *line* in $\mathbb{P}^2$ to be a degree 1 curve. It is left as an exercise to show that any two *distinct* lines in $\mathbb{P}^2$ meet at *exactly* one point.

We now state the main result.

Theorem 2.9. *(Bezout, weak version) Suppose $X = V(f)$, $Y = V(g)$ are curves in $\mathbb{P}^2$ of degree n and m respectively. If f and g share no non-trivial factors, then $X \cap Y$ is finite and $|X \cap Y| \leq nm$.*

We outline a proof given in Toni Annala's notes . Annala's notes also gives the stronger statement which asserts that we have *exactly nm* intersections counted with some appropriately defined multiplicity[5]. The avid reader may wish to fill in the details, although this uses some machinery beyond the scope of this mini-course (in particular, some linear algebra). However, in some sense the result is more important than the proof. It is a remarkable statement: that we have some control on the number of intersections of two curves by some algebraic condition on the polynomials defining them.

Lemma 2.10. *$X \cap Y$ is finite $\iff$ f, g share a non-trivial factor.*

Proof. ($\Longleftarrow$) If h non-trivial factor, any point at which h vanishes (of which there are infinitely many) is in vanishing of f, g.
($\Longrightarrow$) Not hard, but uses some results about $\mathbb{C}[x, y]$. Omitted. □

[5]Defining a 'natural' notion of multiplicity is an interesting problem. One way to do this is *derived algebraic geometry* (ask Sofia for a citation)

Now assume f, g share no non-trivial factors. Let us consider the simpler case of $f, g \in \mathbb{C}[x]$. Let $f = a_n x^n + ... + a_0$, $g = b_m x^m + ... + b_0$. The intersection points are the common roots of f and g. Now f, g have a common factor if and only if f, g share linear factors. This occurs if and only if $af = bg$, for some $a, b \in \mathbb{C}[x]$ non-zero and $\deg(a) < n, \deg(b) < m$ (consider the prime factorisations). Equivalently, there exists constants $c_1, ..., c_n, d_1, ..., d_m \in \mathbb{C}$ such that

$$c_1 f + ... + c_n x^{n-1} f + d_1 g + ... + d_m x^{m-1} g = 0 \tag{7}$$

I claim that this condition is equivalent to some polynomial in the coefficients $a_0, ..., a_n, b_0, ..., b_n$, denoted $Res(f, g)$, vanishing. In the language of linear algebra, we want $f, xf, ..., x^{n-1}f, g, xg, ..., x^{m-1}g$ are linearly dependent. The condition is that $Res(f, g) = \det(Syl(f, g)) = 0$, where

$$Syl(f,g) = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & ... & a_0 & 0 & 0 & ... & 0 \\ 0 & a_n & a_{n-1} & ... & a_1 & a_0 & 0 & ... & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & & ... & & 0 & a_n & ... & a_0 \\ b_n & b_{n-1} & b_{n-2} & ... & b_0 & 0 & 0 & ... & 0 \\ 0 & b_n & b_{n-1} & ... & b_1 & b_0 & 0 & ... & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & & ... & & 0 & b_n & ... & b_0 \end{pmatrix} \tag{8}$$

This can be seen by expanding f and g and comparing coefficients of x^i. $Res(f, g)$ is called the *resultant* of f and g. The exact expression is not important. What we care about is that there exists such a polynomial, and that it can be explicitly calculated.

Proof. (of Bezout) Assume f, g coprime, so $X \cap Y$ is finite.
First we start with some technicalities. By finiteness of $X \cap Y$, there exists a point P such that

1. $f(P), g(P) \neq 0$
2. for any $Q_1, Q_2 \in X \cap Y$, P is not on the line through Q_1, Q_2

Applying a change of coordiantes, without loss of generality, $P = [1 : 0 : 0]$.

Write f, g as polynomials in T

$$f = f_n T^n + f_{n-1} T^{n-1} ... + f_0 \tag{9}$$
$$g = g_m T^m + f_{m-1} T^{m-1} ... + g_0 \tag{10}$$

Exercise: use technicality 1 to show that $f_n, g_m \neq 0$.

Now form the resultant $R = Res(f, g)$ which is a polynomial in the coefficients f_i, g_i, so is a polynomial in x, y. We have $R(x, y) = 0$ if and only if there exists a t such that t is a root of f and g, i.e. $[t : x : y] = 0$ (†).

We finally show that R is a homogeneous polynomial of degree nm. This is done using explicit calculation of the determinant[6]. Showing that it is non-zero is a little subtle[7].

Hence we may write[8] $R(x, y) = \prod_{i=1}^{nm}(a_i x + b_i y)$. Each equation $a_i x + b_i y = 0$ gives a line in $\mathbb{P}^2$ on which, by (†), there is an intersection point. This line goes through $[1 : 0 : 0]$ so condition 2 insures we are counting intersections only once. This proves the result. □

Remark. We lose solutions precisely when the roots of $R(x, y)$ are not distinct. Geometrically, this occurs when the curves do not intersect transversely. Compare this to when we intersected a line with a quadratic and had a root of multiplicity 2.

Remark. As a key takeaway, we have seen that translating geometric problems to algebraic ones is a powerful technique. There is a deep interplay between algebra and geometry, which is starting point of *algebraically geometry.*

3 Lines in a cubic

We end with a brief comment on a significant and well-known example: lines on a cubic surface.

Definition 3.1. An *cubic surface in* $\mathbb{P}^3$ is the zero set of a degree 3 homogeneous polynomial f in four variables

$$V(f) = P \in \mathbb{P}^3 : f(P) = 0$$

[6]$\det A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i\sigma(i)}$ and note that f_i is a homogeneous polynomial of degree $n - i$ and similarly for g.

[7]Uses Gauss' lemma for polynomials

[8]using that R is homogeneous, this follows from the fact that any degree n polynomial over $\mathbb{C}$ in one variable splits completely into linear factors

Question 3.2. *Given a cubic surface, how many lines are contained in it?*

It turns out that for *smooth* cubic surfaces[9] the answer is 27.

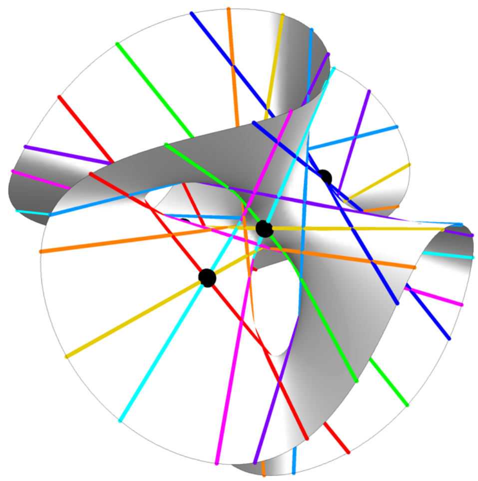

Figure 2. The 27 lines in a cubic

This is miraculous! We have no reason to expect that the answer is finite, let alone that there is the same number of lines in all 'nice' cubic surfaces. The result is specific to cubics. A quadratic surface contains infinitely many lines, and a degree n surface, for $n \geq 4$, contains no lines in general. Standard proofs of the cubic result are advanced, although an excellent and somewhat accessible proof is given in . There are many more results of a similar flavour, some of which require very advanced techniques to solve.

[9]so in particular the surface has no cusps or nodes