Hyperbolic Geometry and the Poincaré Disc Model

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20 July 2021

1 Motivation

In *Euclid's Elements*, Euclid set out five axioms from which he proved various geometric results. They are

1. Two points can be joined by a straight line.
2. Any line segment can be infinitely extended.
3. Given any point and positive radius, we can draw a circle with this point as the centre and this radius
4. All right angles are equal.
5. (*Parallel Postulate*) If two straight lines are intersected by another straight line and the sum of the internal angles on one side of the line is less than $180°$, then the two lines will meet on this side when extended indefinitely.

We may replace the parallel postulate to obtain

5. (*Playfair's Postulate*) Given a straight line l and a point P not on the line, there is exactly one straight line which goes through P and does not intersect l (when both lines are extended indefinitely).

Many geometers tried (and failed) to prove the fifth axiom from the first four. We now know this is not possible, since there are geometries which have the first four axioms but not the fifth. For example, we can consider the two sheeted hyperboloid. We let *lines* on the hyperboloid to be *length minimising curves* and angles between two intersecting lines to be the angle between the tangent vectors there. One may check that the first four axioms hold. One may also show that the fifth axiom is replaced by

5. Given a straight line l and a point P not on the line, there are at least two which goes through P and does not intersect l.

This is an example of *hyperbolic geometry*.

When studying this geometry, one would like a simple description of lines, and also of the *isometries* or distance preserving maps. In this mini-course we will look at the construction of the Hyperbolic (Poincaré) Disc Model and explore some of the isometries.

2 Distances

To begin with, we will revisit the notion of distance and length. We start with a curve γ defined as $y = f(x)$ in the Euclidean (flat) plane, where x takes values in the interval $[a, b]$ and f is differentiable. We partition the interval into sections of length $\triangle x$. Using Pythagoras' theorem, the length of the curve between x and $x + \triangle x$ is approximately $\triangle s^2 = \triangle x^2 + \triangle y^2$, where $\triangle y = f(x + \triangle x) - f(x)$. Summing these lengths up and taking a limit appropriately, we obtain

$$length(\gamma) = \int_a^b \sqrt{1 + \frac{dy}{dx}} dx \tag{2.1}$$

Now consider a more general curve γ defined as $\gamma(t) = (x(t), y(t))$ where x, y are differentiable functions and t takes values in $[a, b]$. Similarly, we obtain that

$$length(\gamma) = \int ds = \int \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\frac{dy}{dt}^2 + \frac{dx}{dt}^2} dt \tag{2.2}$$

We may also change to polar coordinates (r, θ) with $x = r\cos(\theta)$ and $y = r\sin(\theta)$. In this case the length of a curve $\gamma(t) = (r(t), \theta(t))$ is

$$length(\gamma) = \int ds = \int \sqrt{dr^2 + rd\theta^2} = \int_a^b \sqrt{\frac{dr}{dt}^2 + r^2\frac{dr}{dt}^2} dt \tag{2.3}$$

We will not think too deeply here about what expressions like $\int \sqrt{dx^2 + dy^2}$ mean and instead, use this calculation to motivate a more general definition of length.

Definition 2.1. Given a surface with an appropriate parametrisation we define an *abstract Riemannian metric* as an expression of the form $ds^2 = E(u, v)du^2 + F(u, v)dudv + G(u, v)dv^2$. It also must be *positive definite*. This means that $Eu^2 + 2Guv + Gv^2 > 0$ for all $(u, v) \neq 0$ which insures that lengths are positive.

Definition 2.2. Given a surface (with appropriate parametrisation) with a metric $ds^2 = E(u, v)du^2 + F(u, v)dudv + G(u, v)dv^2$ and curve on the surface γ parametrised as $\gamma(t) = (u(t), v(t))$ with t in $[a, b]$, then we define the *length* of γ to be

$$length(\gamma) = \int_a^b \sqrt{E(u, v)\big(\frac{du}{dt}\big)^2 + F(u, v)\frac{du}{dt}\frac{dv}{dt} + G(u, v)\big(\frac{dv}{dt}\big)^2} dt$$

So the Euclidean plane with it's usual cartesian parametrisation, we have $E(x,y) = G(x,y) = 1$ and $F(x,y) = 0$. As an exercise, you may wish to think about how to calculate lengths on a sphere parametrised using spherical polar coordinates and relate it to the discussion above.

Definition 2.3. The *distance* between two points is the "shortest possible" length of curves between the two points.

3 The Hyperbolic Disc

We now have enough definitions to introduce the hyperbolic disc.

Definition 3.1. The hyperbolic disc is the unit disc with centre 0, radius 1 endowed with the abstract Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Intuitively, as we go towards the edge of the disc,

$$ds^2 >> dx^2 + dy^2$$

and so lengths that are the same with respect to the Euclidean metric are the much larger near the edge of the disc. In Escher's illustration of the Hyperbolic disc, shown in Fig. 1, the angels are all the same size with respect to the hyperbolic metric but get smaller as they go towards the edge of the disc with respect the flat Euclidean metric (demonstrated by the red lines). The length of the line from the centre to the edge (shown in green) is infinite.

Given points P and Q in the disc, we wish to know what the curve between them that minimises length is. We first start with the simple case when P is at the origin and Q is on the positive real axis. We wish to minimise

$$Length(\gamma) = \int_a^b \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{x^2 + y^2}} dt$$

This can be achieved by taking $y = \dot{y} = 0$. Hence the length minimising curve is the segment of the diameter which joins P and Q.

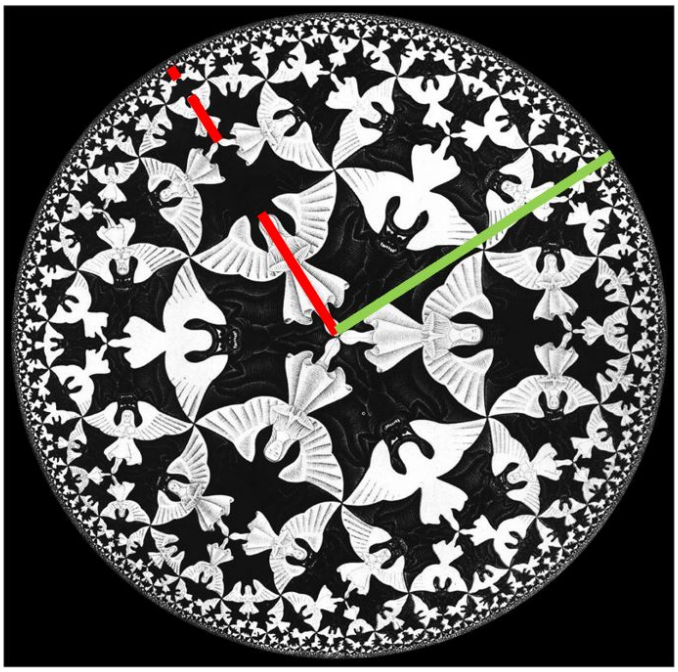

Figure 1: Escher's Circle Limit IV (Heaven and Hell)

We will now to try understand some isometries of the Hyperbolic disc and to do this, we take a small detour to look at Möbius maps.

Definition 3.2. A map $f : \mathbb{C} \to \mathbb{C}$ is a Möbius map if it is of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ and we interpret $1/0 =$ "∞", "∞"/"∞" $= 1$ $1/$"∞" $= 0$.[1]

By writing $z = x + iy$, we can consider a Möbius map to be a transformation of the plane plus a point "at infinity". Some examples of Möbius maps are $f(z) = az, a \neq 0, f(z) = (\cos(\theta) + i\sin(\theta))z, f(z) = 1/z$. These can be thought of as transformations of the plane. Möbius maps have several nice properties. Some examples are given below, with proofs left as an exercise to the reader[2].

1. Möbius maps are continuous bijections. (Moreover they are *smooth*.)

2. Composition of Möbius maps are Möbius maps. (Moreover, they form a group acting on the extended comlex plane.)

[1]More precisely, we consider f to be a function from the *extended complex plane* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A more intuitive picture can be seen by considering the extended complex plane as a projection of the sphere.

[2]1. Note that rational functions are continuous so it suffices to check continuity at ∞ and the preimage of ∞. For bijectivity, proving 2 may help. 2. It is useful to check that the set of all Möbius functions are generated by the transformations $z \mapsto 1/z, z \mapsto az$ where $a \neq 0$ and $z \mapsto z + b$ $b \in \mathbb{C}$. 3. Show that circles and lines are defined by an equation of the form $Az^2 + Bz + C = 0$ and consider their transformations under the generators. 4. Requires some complex analysis, but is essentially the chain rule (and f has non-vanishing derivative at all points).

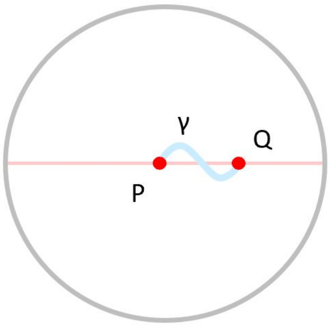

Figure 2: Simple case when P at origin and Q on positive real axis.

3. Möbius maps send circles and lines to circles and lines.

4. Möbius maps preserve angles.

A question we may wish to ask is given a subset S of the plane, what are the Möbius maps which preserve S. When S is the unit disc, some examples would be

1. $f(z) = z$

2. $f(z) = (\cos(\theta) + i\sin(\theta))z$ for $\theta \in \mathbb{R}$

3. $f(z) = \frac{z-1/2}{1-z/2}$

while some non-examples would be

1. $f(z) = z + a$, $a \neq 0$

2. $f(z) = 1/z$

We claim that the set of Möbius transformations preserving the unit discs is

$$\left\{ f(z) = \lambda \frac{z-a}{1-\bar{a}z} \quad | \quad |a| < 1, |\lambda| = 1 \right\} \tag{3.1}$$

This result is useful because of the following theorem.

Theorem 3.3. *Möbius maps which preserve the unit disc are isometries of the hyperbolic disc.*

Proof. For transformations of the form $w = f(z) = \lambda z$ where $|\lambda| = 1$, we have $dw = \lambda dz$. So $|dw| = |dz|$. Also $|z| = |\lambda||w| = |w|$. Note that $ds = |dz|^2/(1-|z|^2)^2$, and the result follows. A similar and more tedius calculation gives invariance of ds under transformations of the form $f(z) = (z-a)/(1-\bar{a}z)$. $\square$

We now return to the question of length minimisers. Consider P and Q, general points in the unit disc. Let P and Q be at the points represented by the complex number a and b respectively. We may apply the transformation $z \mapsto \frac{z-a}{1-\bar{a}z}$. P is mapped to the origin. Applying further a rotation $f(z) = \lambda z$ we can send P to P', Q to Q' where P' is at the origin and Q' is on the positive real axis. Let the composition of these functions be f. We have already seen what the length minimising curve between P' and Q' is - it is the straight line joining the two, call it γ'.

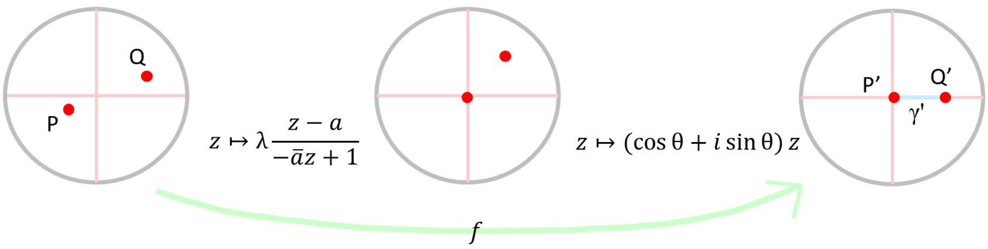

Figure 3: There is a Möbius map sending P to the origin and Q to the positive real axis.

Note that any length minimising curve between P and Q, say γ must satisfy $f(\gamma') = \gamma$ since f preserves all lengths of curves. So $\gamma' = f^{-1}(\gamma)$. Recall that Möbius maps send circles/lines to circles/lines and preserve angles. The real axis is a line which meets the unit circle at right angles. Hence γ will be part of a circle which meets the unit circle at right angles or is a diameter. This motivates our definition.

Definition 3.4. A *hyperbolic line* is a circle which meets the unit disc at a right angle, or a diameter of the unit disc.

We can note several properties of the hyperbolic disc that are like the Euclidean plane.

1. Between any two points there is a hyperbolic line joining them
2. Any finite segment of a hyperbolic line can be extended to an infinite one
3. Can define a *hyperbolic circle* with centre P as the locus of points with distance r away from P

And some more properties which are different from the Euclidean plane.

1. Distinct lines either 1) intersect 2) meet 'at infinity' 3) don't meet at all
2. Internal angles of a hyperbolic triangle add up to less than π
3. Given line R and point P not on R, there are at least two distinct lines through P that do not intersect R

To summarise, we defined the hyperbolic disc as the unit disc with the abstract Riemannian metric $ds^2 = \frac{dx^2+dy^2}{(1-x^2-y^2)^2}$. We found a class of isometries which were Möbius maps preserving the unit disc. We used this to show that the length minimising curves were segments of circles meeting the unit circle at right angles and diameters. This gives rise to a hyperbolic geometry.

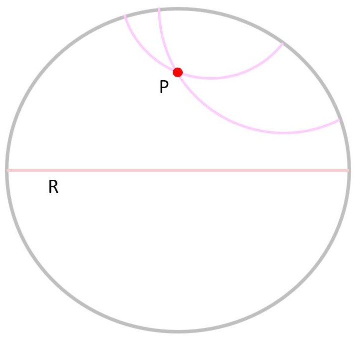

Figure 4: The hyperbolic geometry of the hyperbolic disc.

4 Brief Notes on Extensions and Applications

There are a few ways we can extend our discussion. We saw briefly another model of the hyperbolic space with the two sheeted hyperboloid. Another model is given by the upper half plane equipped with the metric $ds = (dx^2 + dy^2)/y^2$. In fact, all of these models are related. The hyperbolic disc model can be considered as a projection of the hyperboloid model, and the hyperbolic disc model and the upper half plane model are related by the Möbius map $(z - i)/(z + i)$. An analogous higher dimensional *hyperbolic space* can be defined on the n dimensional ball.

Hyperbolic geometry has wide reaching applications, from modular forms to special relatively. For those who are familiar with Minkowski space-time, notes by Barrett[1] goes into detail about the relation with hyperbolic geometry. More generally, non-Euclidean geometry plays a fundamental role in general relatively.

References

[1] Barrett, J.F. *Minkowski Space Time and Hyperbolic Geometry*. Available at: `https://eprints.soton.ac.uk/397637/2/J_F_Barrett_MICOM_2015_2018_revision_.pdf`