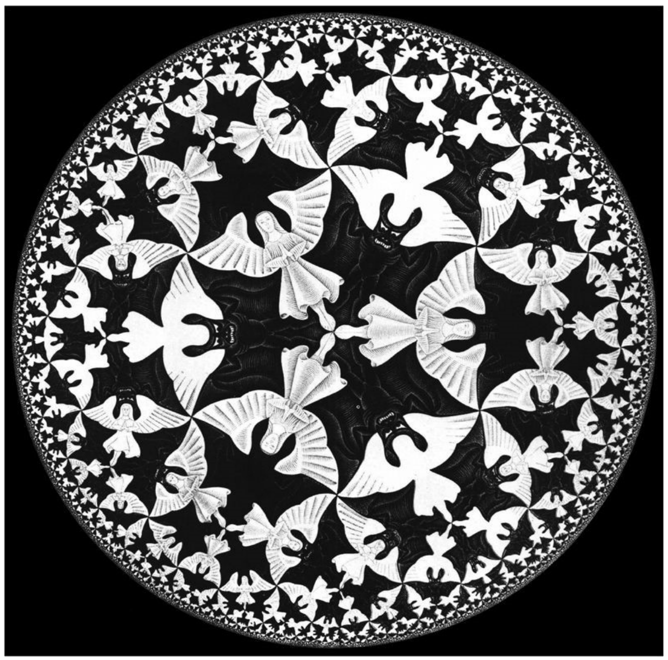


Hyperbolic Geometry and the Poincaré Disc

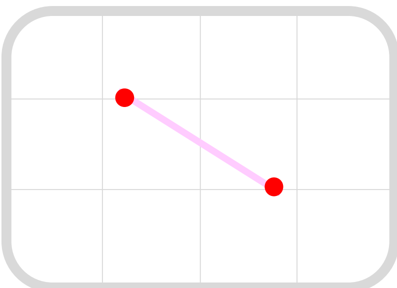
Circle Limit IV (Heaven and Hell) by M.C. Escher

Although the angels/demons look like they are getting smaller as you go towards the edge, they are all, in fact, the same size in the hyperbolic disc model

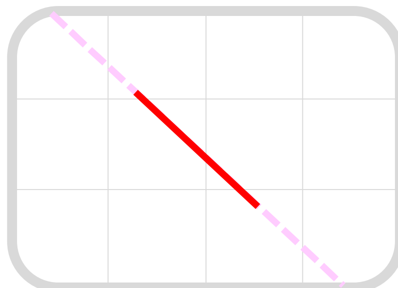
Historical Motivation

Euclid's postulates

1.



2.



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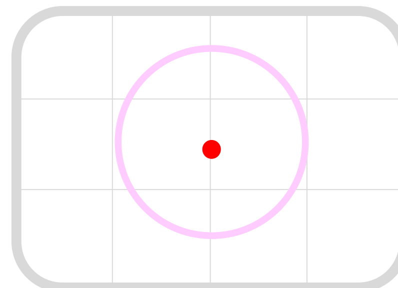

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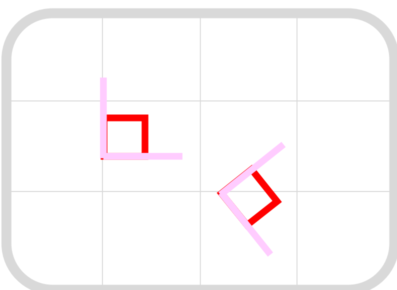


5.

Playfair's axiom, equivalent to the parallel postulate

Historical Motivation

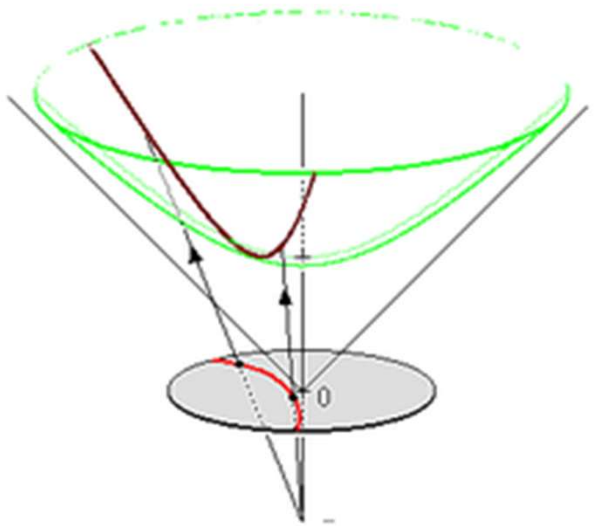
Many geometers tried (and failed) to prove the 5th axiom from the first four

This led to the discovery of **hyperbolic geometry**

We replace axiom 5 with

Given line *R* and point *P* not on *R*, there are **at least two** distinct lines through *P* that do not intersect *R*

Geometry on a hyperboloid



Ref: https://en.wikipedia.org/wiki/Pseudosphere

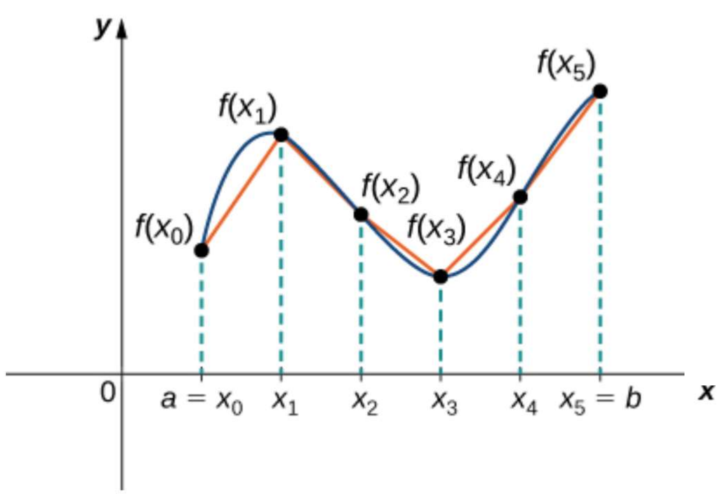
- Lines are curves of minimum length
- Angles between lines are angle between tangent vectors at intersection
- We can convince ourselves that points and lines satisfy axioms 1-4
- Given line *R* and point *P* not on *R*, there are at least two distinct lines through *P* that do not intersect *R*

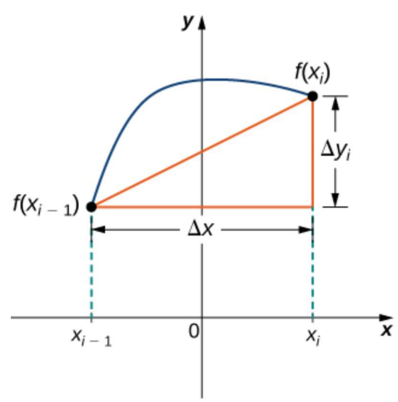
Towards the Hyperbolic Disc

- Want to easily be able to describe lines
- Want the isometries (distance preserving maps) to be simple

Measuring distances in the plane

Q: What is the length of a curve y = f(x) (x in [a,b])?





$$\Delta s^2 \approx \Delta x^2 + \Delta y^2$$

$$\text{Arc length} = \sum \Delta s$$

$$\approx \sum \sqrt{\Delta x^2 + \Delta y^2}$$

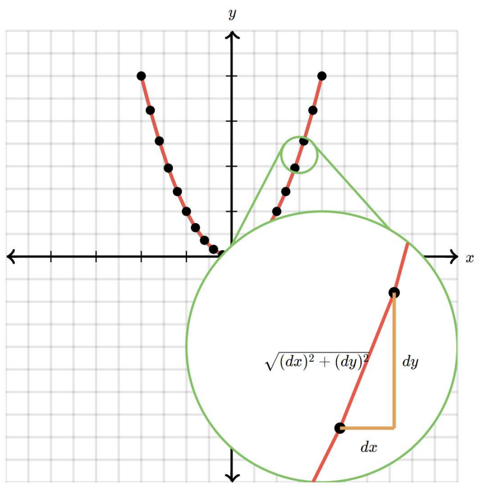
$$\approx \sum \Delta x \sqrt{1 + \Delta y^2 / \Delta x^2}$$

$$\approx \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

Ref:
https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_%28OpenStax%29/06%3A_Applications_of_Integration/6.4%3A_Arc_Length_of_a_Curve_and_Surface_Area

Measuring distances in the plane

Q: What is the length of a curve γ(t) = (x(t), y(t)), t in [a,b]?



Have $ds^2 = dx^2 + dy^2$

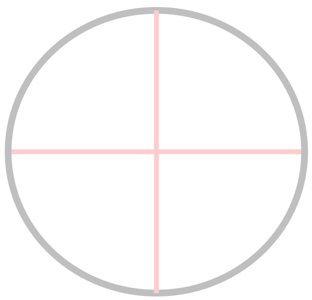
Arc length = $\int ds = \int \sqrt{dx^2 + dy^2}$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ref: https://www.khanacademy.org/math/multivariable-calculus/integrating-multivariable-functions/line-integrals-for-scalar-functions-articles/a/arc-length-part-2-parametric-curve

Hyperbolic Disc Model

The unit disc…



And the (abstract Riemannian) metric …

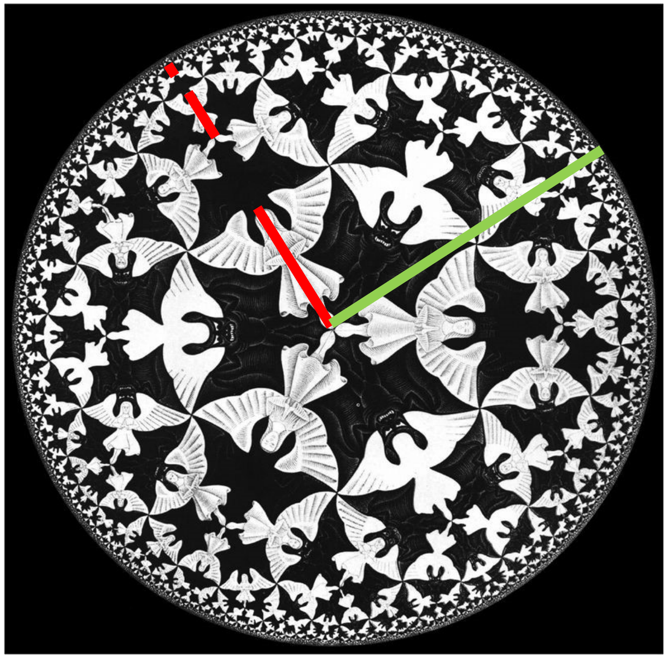
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Also known as Poincaré's disc model.

Historical note: The hyperbolic disc model was proposed by Eugenio Beltrami, and was popularised by Poincaré who rediscovered it 14 years later (ref: https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model)

Intuition

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$



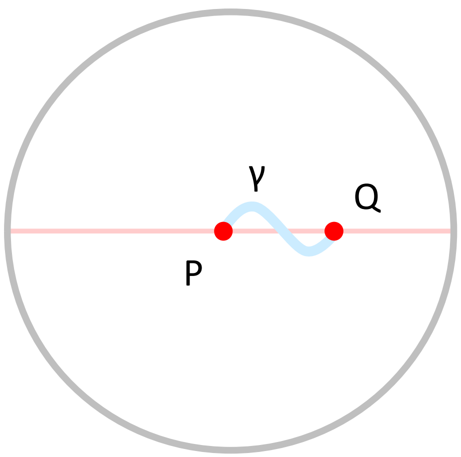
- As you go towards the edge of the disc, $ds^2 >> dx^2+dy^2$
- Lengths that are the same with respect to the normal Euclidean model are much longer near the edge
- The distance from the centre to the edge is infinite

Lines of shortest length

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Given a point P and Q in the unit disc, what is the curve of minimum length between them?

Start with the case P is at origin, Q on positive real axis



Want to minimise $\int \frac{2\sqrt{\dot{x}^2+\dot{y}^2}}{1-x^2-y^2}\,dt$

Make $y = \dot{y} = 0$

Distance Preserving Maps (isometries)

We wish to study distance preserving maps from the unit disc to itself

i.e. Maps f from the unit disc to itself such that if $\gamma(t) = (x(t), y(t))$ is a curve (t in $[a, b]$), then $\Gamma(t) = f \circ \gamma(t)$ has the same length as γ

To do this, we will need to study a class of functions called **Möbius maps**.

Möbius Maps

Möbius maps are complex functions of the form

$$f(z) = \frac{az + b}{cz + d}$$

a, b, c, d complex numbers such that $ad - bc \neq 0$

Where we let $\frac{1}{0} = \infty, \frac{\infty}{\infty} = 1, \frac{\infty}{1} = \infty$ etc.

More precisely, we define Möbius maps from the **extended complex plane** (the complex numbers with the point 'at infinity' added on) to itself

$$f(z) = \frac{az+b}{cz+d}$$

Examples

$f(z) = 2z$

Where a is real

$$f(z) = (\cos\theta + i\sin\theta)\, z$$

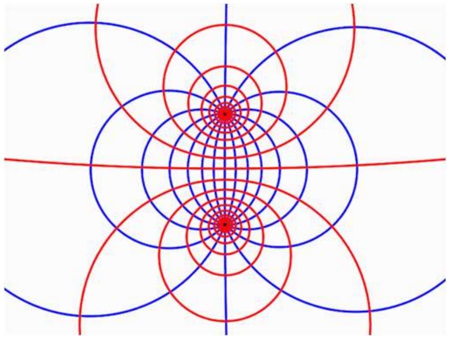
$$f(z) = \frac{az + b}{cz + d}$$

Examples

$$f(z) = \frac{1}{z}$$

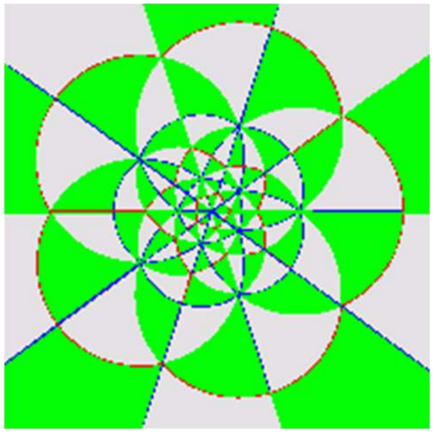
This is left as an exercise to the reader

- Möbius maps are continuous bijections (moreover, they are smooth)
- Compositions of Möbius maps are Möbius maps (they form a group acting on the extended complex plane)
- Möbius maps send lines/circles to lines/circles



This is left as an exercise to the reader

- Möbius maps are continuous bijections (moreover, they are smooth)
- Compositions of Möbius maps are Möbius maps (they form a group acting on the extended complex plane)
- Möbius maps send lines/circles to lines/circles
- Möbius maps preserve angles



Möbius maps preserving the disc

Q: Which Möbius maps send the disc to itself?

Examples	Non-Examples
$f(z) = z$	$f(z) = 2z$
$f(z) = (\cos\theta + i\sin\theta)\, z$	$f(z) = \dfrac{1}{z}$
$f(z) = \dfrac{z - 1/2}{-\frac{z}{2} + 1}$	

Möbius maps preserving the disc

Q: Which Möbius maps send the disc to itself?

A: Möbius maps of the form

$$f(z) = \lambda \frac{z - a}{-\bar{a}z + 1}$$

$|\lambda| = 1$, $|a|<1$

Möbius invariance

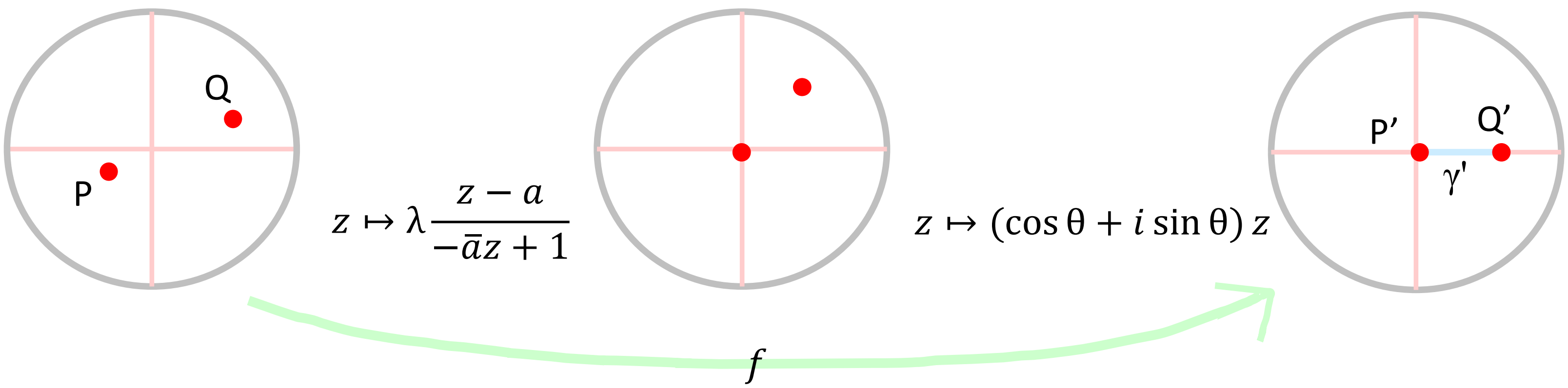
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Claim: The (abstract Riemannian) metric is invariant under Möbius maps

i.e. If $\gamma(t) = x(t) + iy(t)$ is a curve (t in $[a, b]$) and f is a Möbius map preserving the unit disc, then $\Gamma(t) = f \circ \gamma(t)$ has the same length

Length minimisers

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$



The line of minimum length between P' and Q' is γ

Line of minimum length between P and Q is $f^{-1}(\gamma)$

Hyperbolic Lines

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Real axis is a line which meets unit disc at right angle

Recall

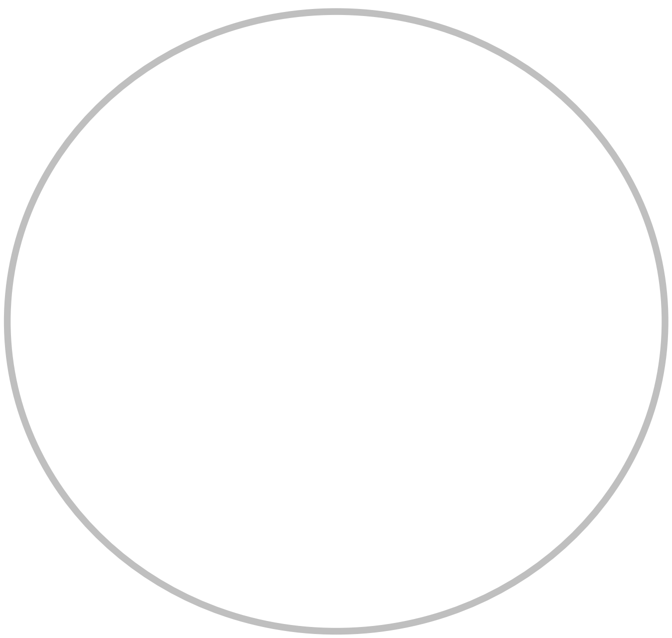
- Möbius maps preserve angles
- Möbius maps send lines/circles to lines/circles

Line of minimum length between P and Q is an **arc of a circle meeting the unit disc at right angles**

These are called **hyperbolic lines**.

Properties

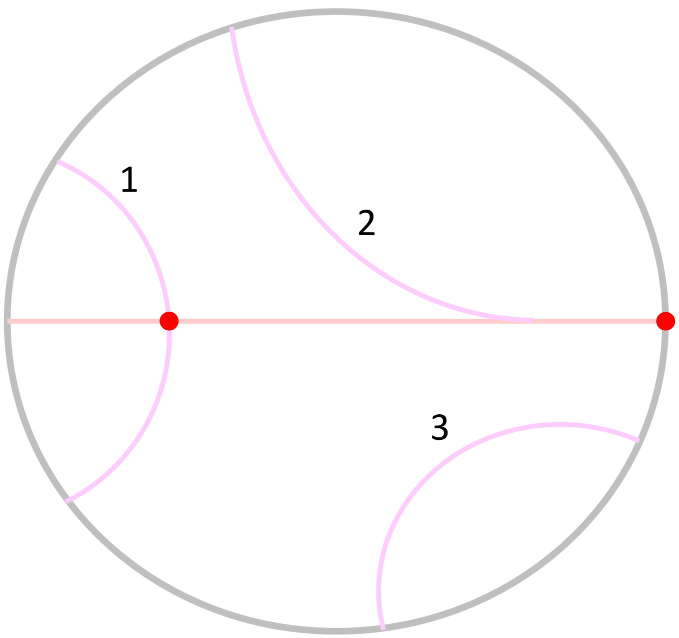
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$



- Between any two points there is a hyperbolic line joining them
- Any finite segment of a hyperbolic line can be extended to an infinite one
- Can define a **hyperbolic circle** with centre P as the locus of points with distance r away from P

Properties

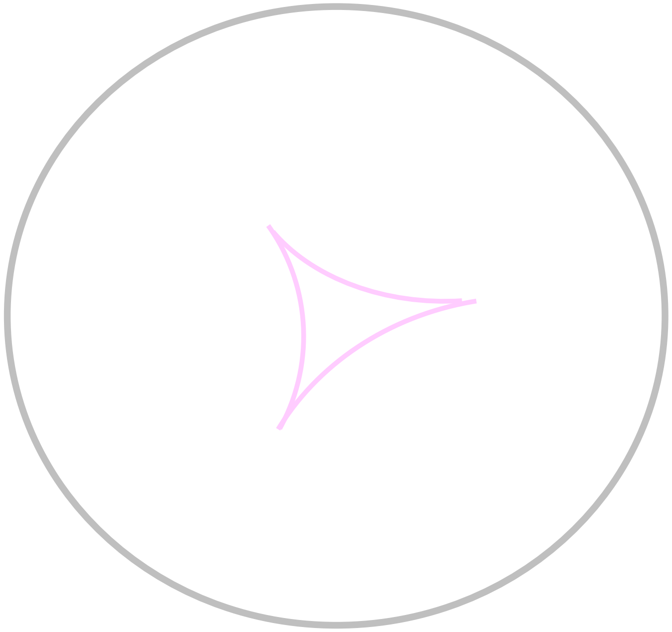
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$



- Distinct lines either
  1) Intersect
  2) Meet 'at infinity'
  3) Don't meet at all

Properties

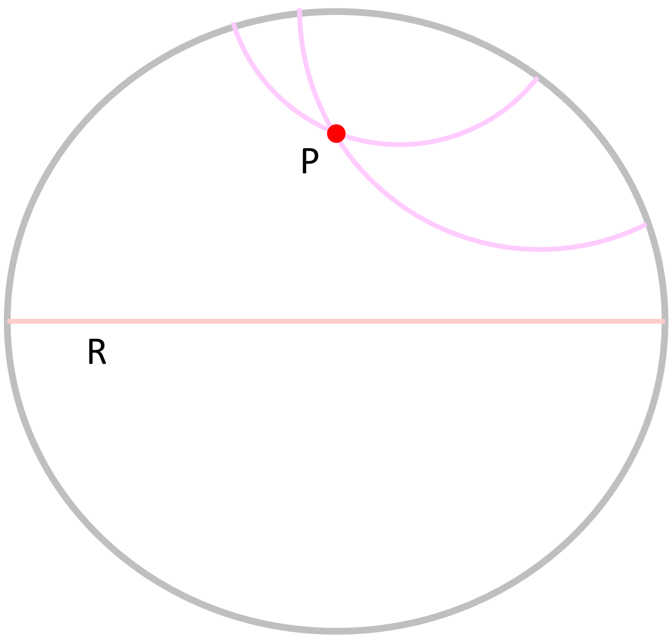
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$



- Distinct lines either
  1) Intersect
  2) Meet ‘at infinity’
  3) Don’t meet at all
- Internal angles of a hyperbolic triangle add up to less than π

Properties

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$



- Distinct lines either
  1) Intersect
  2) Meet ‘at infinity’
  3) Don’t meet at all
- Internal angles of a hyperbolic triangle add up to less than π
- Given line *R* and point *P* not on *R*, there are at least two distinct lines through *P* that do not intersect *R*

Summary

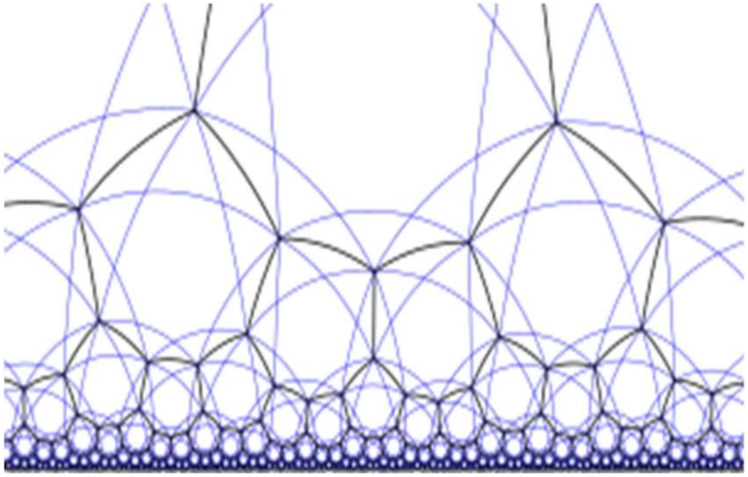
The **hyperbolic disc** is the unit disc + the (abstract Riemannian) metric …

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

- The set of distance preserving maps include **Möbius transformations preserving the unit disc.**
- **Hyperbolic lines** are diameters and circles meeting the unit disc at right angles
- This gives an example of a **non-Euclidean** geometry.

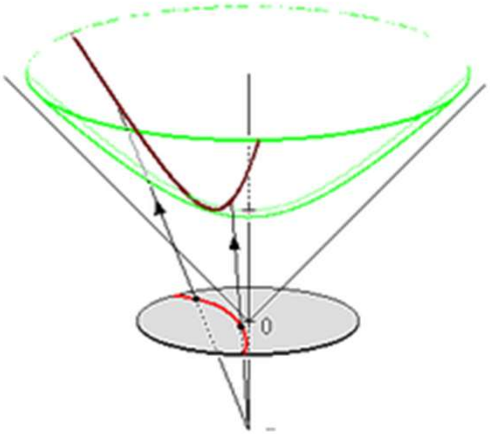
Extensions

- Hyperbolic upper-half-plane model



Ref:
https://en.wikipedia.org/wiki/Poincar%C3%A9_half-plane_model

- Hyperboloid model



Ref:
https://en.wikipedia.org/wiki/Hyperboloid_model

Applications

- Special Relativity

- Modular Forms

Image references
1. https://danielmiessler.com/study/relativity/
2. https://fredrikj.net/blog/2014/10/modular-forms-in-arb/

Further Reading

- Wikipedia for a quick, (mostly) non-technical overview https://en.wikipedia.org/wiki/Hyperbolic_geometry
- The chapter on hyperbolic geometry of Dexter's Notes for a more rigorous approach to abstract Riemannian metrics (has a few prerequisites) https://dec41.user.srcf.net/notes/IB_L/geometry.pdf
- *Minkowski Space Time and Hyperbolic Geometry* by Barrett goes into the connection between Special Relativity and Hyperbolic geometry https://eprints.soton.ac.uk/397637/2/J_F_Barrett_MICOM_2015_2018_revision_.pdf

Möbius invariance

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Claim: The (abstract Riemannian) metric is invariant under Möbius maps

Write $z = x + iy$

Have $ds = 2\frac{|dz|}{1-|z|^2}$

If $w = f(z) = \lambda z$, where $|\lambda| = 1$, then $|w| = |z|$

Also $dw = \lambda dz$ so $|dw| = |\lambda||dz| = |dz|$

Hence $2\frac{|dz|}{1-|z|^2} = 2\frac{|dw|}{1-|w|^2}$

Möbius invariance

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$$

Claim: The (abstract Riemannian) metric is invariant under Möbius maps

If $w = f(z) = \frac{z-a}{-\bar{a}z+1}$, where $|a| < 1$, then

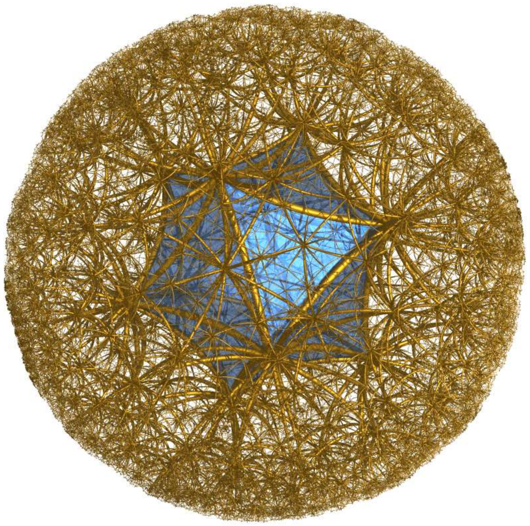
Also $dw = dz(\frac{1}{1-\bar{a}z} + \frac{\bar{a}(z-a)}{(1-\bar{a}z)^2})$

Want to show $2\frac{|dz|}{1-|z|^2} = 2\frac{|dw|}{1-|w|^2}$

Left as exercise to reader☺

Extensions

- Can extend this model to a three dimensional ball
- **Hyperbolic spaces** with 3 or more dimensions



Poincaré 'ball' model view of the hyperbolic regular icosahedral honeycomb
Ref: https://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model