Moduli Spaces of Curves

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In this seminar, I will discuss the moduli space of algebraic curves, then introduce tropical geometry as a method for studying them.

1 Algebraic Curves and Riemann Surfaces

We work over the base field of $\mathbb{C}$. We consider projective space $\mathbb{P}^n = \mathbb{C}^{n+1}/\sim$ where $x \sim \lambda x$ for all $x \in \mathbb{C}^n$, $\lambda \in \mathbb{C} \setminus \{0\}$. Coordinates are given by $[X_0 : ... : X_n]$ for any representative $(X_0, ..., X_n)$ in the equivalence class. We call any set $U_i = \{X_i \neq 0\}$ an *affine patch*. Each affine patch is naturally isomorphic to $\mathbb{C}^n$.

Definition 1.1. A *homogeneous polynomial of degree* d in $\mathbb{C}[X_1, ..., X_m]$ is a polynomial in $\mathbb{C}[X_1, ..., X_m]$ which is a $\mathbb{C}$-linear combination of degree d monomials.

Definition 1.2. The *vanishing locus* of homogeneous polynomials $F_1, ..., F_l$ in $\mathbb{C}[X_1, ..., X_m]$ is $\mathbb{V}(F_1, ..., F_l) = \{P \in \mathbb{P}^n | F_i(P) = 0, i = 1, ..., l\}$

Proposition 1.3. *The vanishing locus is well-defined.*

Proof. We have that $F(\lambda X_0, ..., \lambda X_n) = \lambda^d F(X_0, ..., X_n)$ where d is the degree of F. □

Definition 1.4. A *projective variety* is the vanishing locus of some (finitely many) homogeneous polynomials.

Except for a few special points, locally projective varieties look like $\mathbb{C}^n$. Since they are given by the vanishing of some polynomials, this can be seen via the inverse function theorem. We wish to consider a particular set of 'nice' projective varieties - those that are *smooth*, *irreducible*. By *smooth*, we mean that there are no such special points and by irreducible, the variety doesn't have multiple components. Consequently, such a projective variety *is* in fact a complex manifold of dimension n. Locally, it looks like $\mathbb{C}^n$.

An interesting class of algebraic varieties are *algebraic curves*, which are algebraic varieties of dimension 1. We are working over $\mathbb{C}$, so these are algebraic varieties with real dimension 2. Topologically, they are surfaces, but the complex nature gives us more structure.

Definition 1.5. A *Riemann Surface* is a connected, Hausdorff space R which is locally homeomorphic to an open subset of $\mathbb{C}$, with analytic transition functions.

So an algebraic curve is naturally a Riemann surface. A remarkable result is the converse: that every compact Riemann surface is an algebraic curve.[1] From this point onwards, we will say 'algebraic curve' for smooth, complex, projective, irreducible algebraic curve, used interchangeably with 'Riemann surface'.

Figure 1. Topologically, Riemann surfaces are oriented surfaces

2 Moduli Spaces

Loosely, a moduli space is a space which parametrises objects (often geometric) we are interested in.

Let us start with a simple example. Consider lines in $\mathbb{C}^2$ through the origin. They are parameterised by $\mathbb{CP}^1$. Note we have a topology on $\mathbb{CP}^1$ given by the quotient topology. We can make sense of what is means for two lines to

[1] one way to prove this is through the theory of GAGA

be 'close' to each other. We can also make sense of a continuous *family* of lines, given by a path in $\mathbb{CP}^1$.

Tentative definition *We define $\mathcal{M}_{g,n}$ to be the moduli space of genus g, n marked Riemann surfaces.*

The definition is tentative because we have not formally defined what a moduli space is.[2] This is beyond the scope of this seminar, so we will continue with the informal notion of a space which parametrises isomorphism classes of genus g, n marked surfaces. Often we write $\mathcal{M}_g = \mathcal{M}_{g,0}$.

Example 2.1. Let us find $\mathcal{M}_{0,3}$. This is the space of genus 0 curves with 3 points. We assert that every Riemann surface of genus 0 is isomorphic to $\mathbb{P}^1$, also known as the *Riemann sphere* $\mathbb{C}_\infty$. The automorphisms are precisely the Möbius transformations.[3] The group of Möbius transformations acts *triply transitively* on the Riemann sphere. So given a pair of three points, there is a unique automorphism taking one to the other. Hence there is exactly one genus 0 curve with 3 points up to isomorphism. $\mathcal{M}_{0,3}$ = point. (The uniqueness of the map is important in the definition of a moduli space.)

Example 2.2. Let us find $\mathcal{M}_{0,4}$. As above, we use that the unique Riemann surface of genus 0 is the sphere and that it's automorphism group consists of Möbius transformations. If we have four distinct points on $\mathbb{P}^1$, say (p_1, p_2, p_3, p_4), then as above we may apply a (unique) automorphism such that the first three points are mapped to $0, 1, \infty$. So a genus 0, 4 marked Riemann surface is specified by where the fourth point is sent. In other words

$$\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$$

As an aside, there is an explicit way to calculate this point, given by the *cross ratio* of $p_1, ..., p_4$.

Example 2.3. A slightly harder example. We consider $\mathcal{M}_{1,1}$, the moduli space of genus 1 algebraic curves marked with one point, otherwise known as an *elliptic curve*. It is known that any such curve is given as a quotient $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$. Without loss of generality, $\Lambda = \langle 1, \tau \rangle$, with τ in the upper half plane. For two lattices Λ and Λ', $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ if and only if the two lattices are related by a Möbius transformations $f(z) = (az+b)/(cz+d)$ with $a, b, c, d \in \mathbb{Z}$. So $\mathcal{M}_{1,1}$ is the upper half plane quotiented by $SL_2(\mathbb{Z})$.

[2] The definition is secretly a theorem. A significant result is that such a 'space' (more accurately, a stack) exists

[3] $f(z) = \frac{az+b}{cz+d}$ with suitable extension to infinity.

Remark. $\mathcal{M}_{g,n}$ is not usually compact. We often compactify it to form $\overline{\mathcal{M}}_{g,n}$.

Why moduli? Moduli spaces of algebraic curves are one of the most studied spaces in algebraic geometry. They are complicated enough to be interesting, but are simple enough that we have a handle on their properties. However, even if solely interested in individual algebraic curves, one naturally arrives at moduli spaces.[4]

3 Tropical Curves

We introduce tropical curves[5] as a degeneration of algebraic curves. The intuitive idea is that algebraic curves are somewhat squishy, while tropical curves are incredibly rigid. There are many more things we can say about tropical curves, and many of those things can be used in turn to deduce properties of algebraic curves.

We make this correspondence more explicit. Consider an algebraic curve C given by a degree d homogeneous polynomial in $\mathbb{P}^2$ and restrict to $(\mathbb{C}^*)^2$. Now apply the map

$$Log : (x, y) \mapsto (\log|x| + \log|y|) \tag{1}$$

The result is called the *amoeba* of the curve. We have d tendrils going off to infinity in the directions $(1, 1), (0, -1), (-1, 0)$, with some structure in the middle. The number of holes in the amoeba gives the genus of the algebraic curve (not obvious). To make the structure clearer, we take a family of such amoebas

$$Log_t : (x, y) \mapsto (-\log_t|x| + -\log_t|y|) = \left(-\frac{\log|x|}{\log t}, -\frac{\log|y|}{\log t}\right) \tag{2}$$

and take the limit as $t \to 0$. The resulting skeleton is the *tropical curve* determined by C.[6]

Tropicalisation is a *very* powerful technique, and a very new technique. For instance, there is a tropical analogue of Bezout's theorem on intersection of

[4]For example, one proof that there are 27 lines in a cubic involves considering the moduli space of pairs (l, C) where l is a line contained in a cubic C, and considering the projection maps to the moduli space of cubics and lines respectively.

[5]so named because of Brazilian mathematician Imre Simon

[6]while this is the intuitive method for doing this, in practicality this is rather awful. Alternatively, use *Puiseux series*

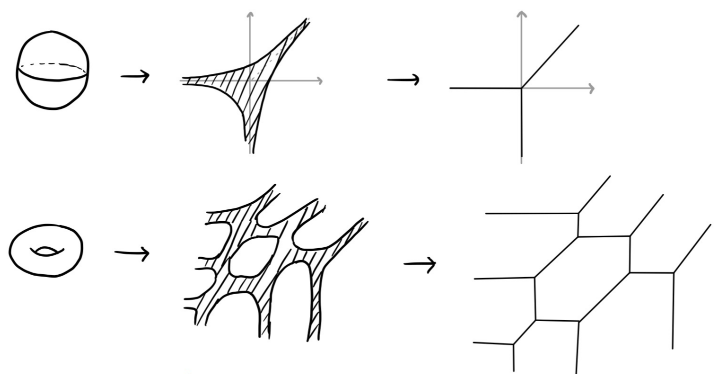

Figure 2. Algebraic curves to amoebas to tropical curves

curves. By a correspondence theorem, this can be used to prove classical Bezout's theorem. New results have also been proven using tropical techniques. However, the relation between algebraic and tropical curves is not fully understood at all. Many features of algebraic curves have tropical analogues, such as divisors and the Riemann-Roch theorem, but it is not clear if they are related.

4 Moduli Space of Tropical Curves

In this final section, we will look at the moduli space of abstract tropical curves and relate this back to the moduli space of algebraic curves. Firstly, observe that a tropical curve can be considered as a *metric graph*, but also allowing infinite edges. This motivates the next definition.

Definition 4.1. An *abstract* tropical curve is a connected metric graph (graph where each edge is given a positive length), which may have leaves with infinite length.

Remark. Tropical curves are embeddings of abstract tropical curves into $\mathbb{R}^n$, given by piecewise linear functions on an abstract tropical curve. An interesting question is how the space of embeddings of tropical curves into $\mathbb{R}^n$ is related to the space of embeddings of algebraic curves into $\mathbb{P}^n$.

Two metric graphs are considered the same if they give the same metric space. We say an abstract tropical curve has *genus* g if it has g holes.[7]

[7] We can formalise this by saying it has 1st Betti number g

We wish to construct $M^{trop}_{g,n}$, the moduli space of abstract tropical curves of genus g with n leaves. Given an abstract tropical curve, we can deform it by changing the edge lengths. This gives a d dimensional cones $\{x \in \mathbb{R}^d : x_i \geq 0\}$, where d is the number of edges with finite length, quotienting by automorphisms as necessary. When an edge length hits 0, it can then be deformed into a graph with a different configuration of edges.

However we have a problem. Suppose we have an abstract tropical curve with a loop. If all the edges are shrunk to length 0, then we lose a hole. To fix this, we add integer *weights* to the vertices. We then define the genus to be the sum of the number of holes in the graph and the weights. We now glue the different parts together to form our moduli space $M^{trop}_{g,n}$. This is an example of a *generalised cone complex*.

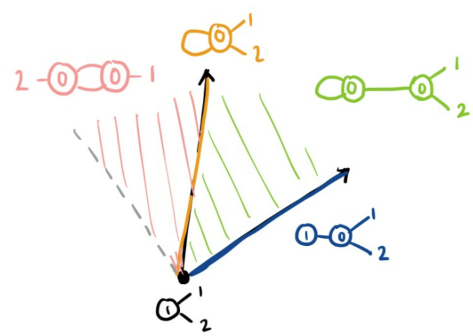

Figure 3. A diagram of $M^{trop}_{1,2}$

We return to the moduli space of abstract tropical curves. Recall that $\mathcal{M}_{g,n}$ is not usually compact. To see why this happens, consider the one pointed torus. One way to deform the torus is to shrink the size of the tube at a given point.[8] However, when the size of the tube hits 0, we no longer have a smooth curve, but a curve with a nodal singularity.

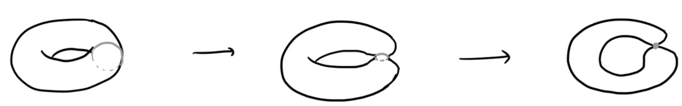

Figure 4. Deforming a complex torus to form a node

[8] The *pair of pants* construction is enlightening

Definition 4.2. A *stable* curve is an algebraic curve whose only singularities are nodes and whose automorphism group is finite. In particular, any component of genus 0 has at least three special points (singularities or marked points) and any component of genus 1 has at least 1 special point.

$\overline{\mathcal{M}}_{g,n}$ is the moduli space of genus g, n marked stable curves.

Now for a stable curve C, we can form a *dual graph* with vertices v corresponding to irreducible components C_v, with an edge between them for each node in $C_v \cap C_w$

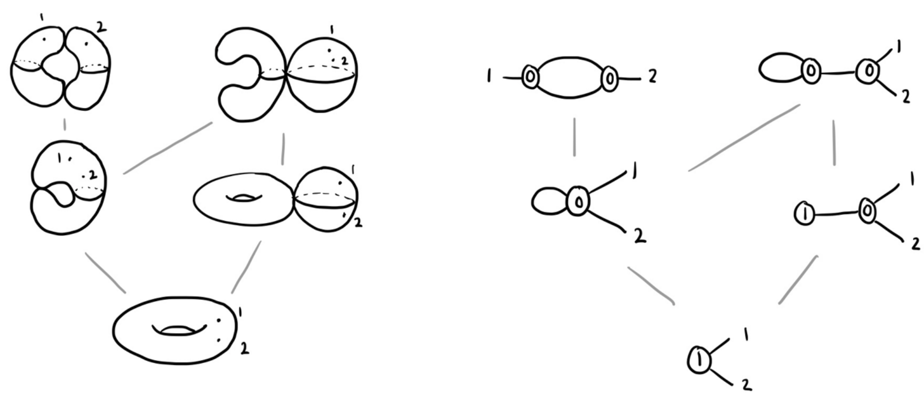

Figure 5. Strata of $\overline{\mathcal{M}}_{g,n}$ and the corresponding dual graph

The dual graphs are abstract tropical curves without the edge lengths. In fact, we may arrive at abstract tropical curves by adding edge lengths to the graphs found in this way. We will end here, but further reading can be found at `https://www.math.brown.edu/mchan2/Mg.pdf`.