University of Cambridge

Singular Points on Hilbert Schemes of Points

Part III Essay

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CONTENTS

INTRODUCTION

Loosely, a moduli space is a space parameterising some geometric objects. Moduli spaces are of significant interest. Even when studying single geometric objects, it is often useful to consider an appropriate moduli space, and there are many interesting geometric questions we can ask about them. An important class of examples are *Hilbert schemes*, which parametrise the closed subschemes of $\mathbb{P}^n$, and more generally the subschemes of a fixed scheme X, with given *Hilbert polynomial* P. The *Hilbert scheme of points*, denoted $\mathrm{Hilb}^d(X)$, are those which parametrise the zero dimensional subschemes of length d.

Hilbert schemes of points have many useful features. They represent a certain moduli functor, so are *fine moduli spaces* (see Definition 1.1.3), which allows the tangent space to be described via deformations. In general, Hilbert schemes are quasi-projective (due to Grothendieck, see Theorem 1.2.3) and connected (due to Hartshorne [17], also see Proposition 1.4.3). The Hilbert scheme of points of a *surface* is particularly well-behaved. Fogarty [11] proved that for a non-singular surface S, $\mathrm{Hilb}^d(S)$ is non-singular of dimension $2d$. For $\mathrm{Hilb}^d(S)$, the cohomology has been studied [12, 24, 32] and enumerative geometry results derived [2, 26].

However, for general X, the Hilbert scheme of points is highly pathological. Vakil [33] defines a moduli space for which *Murphy's Law applies* as those on which every singularity type of finite type over $\mathbb{Z}$ appears, and Jelisiejew [20] shows that for $\mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^n) = \amalg_d \mathrm{Hilb}^d(\mathbb{A}^n)$, with $n \geqslant 16$, *Murhpy's Law holds up to retraction*. This describes a kind of "arbitrarily bad behaviour". Concretely, for a smooth variety X of dimension n, $\mathrm{Hilb}^d(X)$ always has a *smoothable* component of dimension dn (see Section 2.1). Iarrobino [19] showed that for a non-singular variety V of dimension $n \geqslant 3$, $\mathrm{Hilb}^d(V)$ is reducible for d sufficiently large, and demonstrated families that are too large to fit in the smoothable component. Cartwright–Erman–Velasco–Viray [5] studied $\mathrm{Hilb}^8(\mathbb{A}^n)$ and found components of lower dimension for $n \geqslant 4$.

Despite this rather terrifying situation, light can be shed. For $\mathrm{Hilb}^d(\mathbb{P}^n)$, we have a bound on the dimension of the tangent space at the most singular point due to Briançon–Iarrobino [4]. Fix n and represent closed subschemes of $\mathbb{P}^n$ by ideals $I \subseteq k[x_0, \dots, x_n]$. If $[I]$ is a point on $\mathrm{Hilb}^d(\mathbb{P}^n)$ with tangent space $T(I)$ of maximal dimension, then as d grows, $\dim_k T(I)$ grows as $\mathcal{O}(d^{(2-2/n)})$ (big O notation, see Theorem 3.0.1 for precise statement).

To show this result, we reduce to *Borel fixed ideals*, which are in particular monomial ideals (see Section 3.1), then relate these to *generalised Young diagrams*. This motivates the following question.

Q: What are the most singular points of $\mathrm{Hilb}^d(X)$?

The following was conjectured by Briançon and Iarrobino in 1978.

Conjecture A (Briançon–Iarrobino [4])**.** *The ideal* $\mathfrak{m}^r = (x_0, \dots, x_n)^r$ *has the maximum dimension tangent space among all points in* $\mathrm{Hilb}^{\binom{n+r-1}{n}}(\mathbb{P}^n)$.

Little progress has been made on this conjecture. Ramkumar–Sammartano [27] prove a partial result for $\mathrm{Hilb}^d(\mathbb{P}^3)$ by decomposing the tangent space using *signatures* (see Definition 3.0.2). Rezaee [29] obtains more refined conjectures for $\mathrm{Hilb}^d(\mathbb{P}^3)$, including a necessary condition on I for the tangent space $T(I)$ to have maximal dimension which implies Conjecture A (see Conjecture 3.4.7). Bejleri–Stapleton [3] use similar techniques to prove the analogous result for *punctual Hilbert schemes*, which parametrises the "fat" points of the Hilbert scheme of points. The aim of this essay is to present some of these results.

0.1. **Motivation and brief survey.** The smoothness of $\mathrm{Hilb}^d(S)$ for non-singular surface S has far-reaching consequences. Göttsche [12] proves that the Euler and Betti numbers of $\mathrm{Hilb}^d(S)$ depend only on S and gives a generating function for both. A surprising application of this formula is in enumerative geometry. A *K3 surface* is a compact, simply connected complex manifold of dimension 2 with nowhere vanishing holomorphic 2-form [26]. K3 surfaces are Calabi–Yau, and have applications in string theory and mirror symmetry. A fundamental question regarding K3 surfaces is: how many rational curves lie on it? Let β_g be a primitive curve class on a smooth projective K3 surface S with $\beta_g^2 = 2g-2$, and N_g the number of rational curves in class β_g. Yau–Zaslow [35] predicts N_g to satisfy the following generating series

$$\sum_{g \geqslant 0} N_g q^{g-1} = \frac{1}{\triangle(q)},$$

where $\triangle(q) = q\prod_{m\geqslant 1}(1-q^m)^{24}$ is the modular form of weight 12. This was proven by Beauville [2] by relating the coefficients N_g to Göttsche's formula for the Euler number $e(\mathrm{Hilb}^g(S))$. More recently, Oberdieck [26] expanded on these results by considering the enumerative geometry of $\mathrm{Hilb}^d(S)$ for S a K3 surface. A nice account and some further applications can be found in [13].

Another application is in representation theory and combinatorics. The *Macdonald positivity conjecture* is an important result related to *Macdonald polynomials*, which was proven by Haiman [15] via a combinatorial result now known as *Haiman's* $n!$ *theorem*. Haiman uses the *isospectral Hilbert scheme* which is a reduced fibre product

$$\begin{array}{ccc} X_n & \longrightarrow & (\mathbb{C}^*)^n \\ \downarrow & & \downarrow \\ \mathrm{Hilb}^n(\mathbb{A}^2) & \xrightarrow{\varphi} & S^n(\mathbb{A}^2) \end{array}$$

where φ is the *Hilbert–Chow morphism* (see Section 1.4). The smooothness of $\mathrm{Hilb}^n(\mathbb{A}^2)$ is essential in the proof. A related result is Haiman's $(n+1)^{n-1}$ theorem. An exposition of both is given in [22].

For higher dimensional X, the picture for $\mathrm{Hilb}^d(X)$ remains much more obscure. Much work has been done to understand the different components that may arise [8, 21, 28]. Jilisiejew makes a summary of various known components other than the smoothable one in [21, Section 5.6]. Cartwright–Erman–Velasco–Viray [5] show that $\mathrm{Hilb}^d(\mathbb{A}^n)$ is irreducible for $d < 8$, and for $d = 8$ irreducible if and only if $n \leqslant 3$. More results for the reducibility/irreducibility of $\mathrm{Hilb}^d(\mathbb{A}^n)$ are collated by Douvropoulos–Jelisiejew–Nødland–Teitler [8] and the bound improved. It is known that $\mathrm{Hilb}^d(\mathbb{A}^n)$ is (i) irreducible for $d \leqslant 7$ and all n, as well as $(n, d) = (3, 9), (3, 10), (3, 11)$; and (ii) reducible for all $n \geqslant 3$ and $d \geqslant 78$, or $d \geqslant 8$ and $n \geqslant 4$. The only cases that remain unknown are $n = 3$ and $12 \leqslant d \leqslant 77$. These results extend to $\mathrm{Hilb}^d(V)$ for V a general irreducible variety of dimension n. Whether $\mathrm{Hilb}^d(\mathbb{A}^n)$ is reduced has also been considered [20, 31], but only isolated cases are known.

Vakil [33] suggests that unless there is a good reason for a moduli space to be well-behaved, it will be very badly behaved. This heuristic applies to the Hilbert scheme of points, as demonstrated by Jelisiejew's result on Murphy's law. Hence there is no possibility of constraining the singularities of $\mathrm{Hilb}^{\mathrm{pts}}(X)$ for general X. However, Briançon–Iarrobino's bounds suggest that the maximally singular points grow with d in a controlled way. One would hope that if these points could be understood, then there will be a wider range of applications available. Understanding the tangent space of $\mathrm{Hilb}^d(\mathbb{A}^n)$ reduces to a combinatorial problem by standard techniques, and then the connection to Young diagrams used by Haiman in the proof of the $n!$ theorem is a promising direction. This is the method considered in [3, 27, 29], and the one this essay will focus on.

0.2. **Outline.** In Section 1 we define moduli spaces and introduce the Hilbert scheme of points. In Section 2 we discuss some features of the Hilbert scheme of points and the pathologies that may arise. We discuss Jelisiejew's result that Murphy's law holds up to retraction for some Hilbert scheme of points. In Section 3 we discuss the tangent space in more detail. We reduce to monomial ideals, then use Young diagrams to prove Briançon–Iarrobino's bounds. We present an alternative proof of Fogarty's result for smoothness and discuss progress on the conjecture of maximally singular points. Finally, in Section 4, we present Bejleri–Stapleton's analogous result for the punctual Hilbert scheme.

Throughout, we work over a field k, assumed to be algebraically closed of characteristic 0, although many results will hold more generally. We will often denote $R = k[x_1, \dots, x_n]$, $S = k[x_0, \dots, x_n]$, and the maximal ideal $\mathfrak{m} = (x_0, \dots, x_n)$ or $(x_1, \dots, x_n)$.

Acknowledgments. The author thanks Fatemeh Razaee for setting the essay title, and for insightful guidance.

1. Basic Definitions and First properties

1.1. **Moduli functors and moduli spaces.** We define moduli spaces, mainly following definitions from [16, Section 1].

Firstly, let us illustrate the principle with an example. Recall that $\mathbb{RP}^1$ parametrises lines through the origin in the plane. The topology of $\mathbb{RP}^1$ has information about which lines are "close together". We can "survey" this topology by considering a topological space B and a continuous map $f : B \to \mathbb{RP}^1$. If we fix such a map f, then for each point $b \in B$ we obtain a line corresponding to $f(b)$. Such an allocation of lines to each point of B is a *family of lines over a base* B. For instance, if $B = [0, 1]$ is the unit interval, then this allows us to have a notion of a "path of lines".

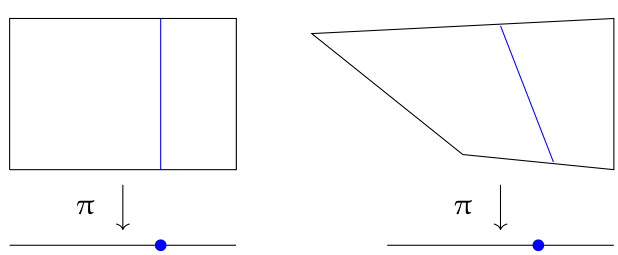


Figure 1. Two families of lines over $B = [0, 1]$.

Another way to view such a family of lines is as a topological space $X \subset B \times \mathbb{R}^2$ and $\pi : X \to B$, such that π is the projection to B restricted to X, and for all $b \in B$, $\pi^{-1}(b)$ is a line through the origin in $\mathbb{R}^2$. Note, we have a *universal family*, which is the tautological line bundle over $\mathbb{RP}^1$. This is universal in the sense that all families of lines can be uniquely realised as the pullback diagram:

$$\begin{array}{ccc} X & \longrightarrow & E \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_{taut}} \\ B & \xrightarrow{f} & \mathbb{RP}^1 \end{array}$$

By Yoneda's lemma, the information of all families of lines, i.e. all maps $f : B \to \mathbb{R}P^1$, determines $\mathbb{RP}^1$ up to unique isomorphism. In this spirit, *a moduli space $\mathcal{M}$ (if it exists) is determined by a suitable notion of families of objects*. Moreover, moduli space $\mathcal{M}$ parametrising some geometric objects should have the following features:

(1) The points of $\mathcal{M}$ are in bijective correspondence with (equivalence classes of) the objects of interest.
(2) The topology of $\mathcal{M}$ encodes which objects are "close together".

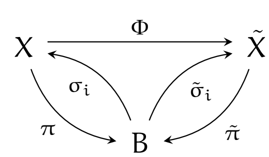
A *moduli problem* consists of (i) a class of objects; (ii) a notion of a family of objects over B; (iii) a notion of equivalence of families. The set of all families over B is denoted $S(B)$. We are purposefully vague with what we mean by "a class of objects" or "a notion of equivalence" since there is a broad range of moduli problems we may wish to consider.

Example 1.1.1 (The Grassmanian)**.** Generalising the first example, we can consider families of the form $\pi : X \to B$, where $X \subseteq B \times \mathbb{R}^r$, π is the projection to B, and for each $b \in B$, $\pi^{-1}(b)$ is a linear subspace of dimension ℓ. Two families are equivalent if they are the same. The corresponding moduli space is the Grassmanian $\mathrm{Gr}_\ell(\mathbb{R}^r) = \mathrm{Gr}_\ell(r)$.

Example 1.1.2 (The moduli space of curves)**.** The *moduli space of genus g, n-pointed curves* $\mathcal{M}_{g,n}$ is constructed[1] by considering families of the form

$$(X, B, \pi, \sigma_1, \dots, \sigma_n),$$

where $\pi : X \to B$ is a map[2] such that $\pi^{-1}(b)$ is a smooth, projective, connected, (arithmetic) genus g algebraic curve for all $b \in B$, and $\sigma_i : B \to X$ are disjoint sections of π. Two families, $(X, \pi, \sigma_1, \dots, \sigma_n)$ and $(\tilde{X}, B, \tilde{\pi}, \tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$, are equivalent if there is an isomorphism $\Phi : X \to X'$ such that the following diagram commutes:



Moduli spaces of curves are well-studied. Some excellent expository pieces are Cavilieri's lecture notes [6] on pointed rational curves and their characteristic classes, and Chan's article [7] on using tropical techniques to calculate cohomology.

Although the notion of moduli functors extends to other categories, we will be working over k-schemes. A *moduli functor* is a functor $\mathcal{F} : \mathbf{Sch} \to \mathbf{Set}$ which sends an object B to $S(B)/\sim$, the set of families over B modulo equivalence. Recall that a functor $\mathcal{F}$ is *representable* by $\mathcal{M}$ if $\mathcal{F}$ is naturally isomorphic to the functor $B \mapsto \mathrm{Mor}_{\mathbf{Sch}}(B, \mathcal{M})$.

[1] Strictly speaking, the (fine) moduli space of curves is a *stack* or alternatively one can consider a *coarse* moduli space, but neither notions are required here.

[2] Flat and proper.

Definition 1.1.3. [16, Definition 1.1]. If $\mathcal{F}$ is representable by a scheme $\mathcal{M}$, say $\mathcal{M}$ is a *(fine) moduli space* for the moduli functor $\mathcal{F}$.

If a moduli space $\mathcal{M}$ exists, then it has many desirable properties:

(1) Since the set of families over a point are the objects of interest up to equivalence, the k-points of $\mathcal{M}$ classify objects of the moduli problem up to equivalence.
(2) (Universal property). Consider the morphism $\mathrm{id} : \mathcal{M} \to \mathcal{M}$. Via the isomorphism of functors defining $\mathcal{M}$, we obtain an element of $\mathcal{F}(\mathcal{M})$, i.e. a family over $\mathcal{M}$. This is the *universal family* for the moduli problem. Every family arises as a pullback the universal family.

Remark 1.1.4. (Philosophical interlude). Why do we care about moduli spaces? For one, they are spaces which have rich structure but are "nice enough" for geometric questions to be tractable. They form examples of classes of spaces we may be interested in - for instance, the Hilbert scheme of points of K3 surfaces are some of the only known examples of holomorphic-symplectic varieties [26]. Even if one is only interested in the geometric objects themselves, considering their moduli spaces is natural. In particular, many enumerative problems can be solved by considering incidence correspondence and projections to known moduli spaces: see [34, Chapter 27] for 27 lines in a cubic (a classic example), or [1] for the interpolation problem (a more recent application).

1.2. **Hilbert polynomials and Hilbert schemes.** Hilbert schemes parametrise closed subschemes of $\mathbb{P}^r$, or more generally, closed subschemes of a fixed scheme X. However, the moduli problem considering all families of such subschemes is not representable. It turns out that a sensible thing to do is to consider flat families with fixed Hilbert polynomial.

Recall that an A-module M is *flat* if the functor $M \otimes_A -$ is an exact functor. A morphism $\varphi : X \to Y$ of schemes is *flat at* $p \in X$ if the stalk $\mathcal{O}_{X,p}$ is a flat $\mathcal{O}_{Y,\varphi(p)}$-module, and *flat* if it is flat at p for all $p \in X$.

Flatness is a kind of "regularity condition" on the fibres of a morphism of schemes. It can be shown that in reasonable cases[3] many important invariants such as degree, dimension, and arithmetic genus are constant along the fibres $X_y = \varphi^{-1}(y)$ of a flat morphism $\varphi : X \to Y$. There is more on flatness in [34, Chapter 24].

Lemma 1.2.1. *A morphism of affine schemes* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is flat if and only if the corresponding morphism of rings* $A \to B$ *is flat.*

Proof. Flatness can be checked stalk locally. □

Given a subscheme $X = \mathbb{V}(I)$ of $\mathbb{P}^r$, we can associate to it a *Hilbert polynomial* P. By definition, $P(m)$ is the dimension of the degree m piece of the graded k-algebra S/I (as a k vector space) for large m. An introduction to Hilbert polynomials is given in [9, Section 1.9]. The Hilbert polynomial encodes various geometric properties of X:

(1) The degree n of P is the dimension of X.
(2) The leading coefficient of P multiplied by $n!$ is the degree[4] d of X.

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[3]Specifically, $\varphi : X \to Y$ projective, flat morphism with Y Noetherian and connected.

[4]The degree of a subscheme is the generic number of intersections with a hyperplane. The clash in terminology is unfortunate.

Proposition 1.2.2. *Suppose* B *is a Noetherian scheme, and* $X \subset \mathbb{P}^r \times B$ *a closed subscheme. Let* $\pi : X \to B$ *be the natural projection. For each* $b \in B$, X_b *can be considered as a closed subscheme of* $\mathbb{P}^n_{\kappa(b)}$ *with Hilbert polynomial* P_b. *If* π *is flat and proper, then* P_b *is independent of* b.

Proof. See [18, III Theorem 9.9] or [34, Corollary 24.7.1]. □

The functor $\mathbf{Hilb}_{P,r}$ sends a Noetherian scheme B to the set of families of the form:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ i\ } & \mathbb{P}^r \times B \\ & \searrow^{\varphi} & \downarrow \pi_B \\ & & B \end{array}$$

Here π_B is the projection, i is a closed immersion, φ is proper and flat, with fibres $\mathcal{X}_b$ having fixed Hilbert polynoimal P.

Theorem 1.2.3 (Grothendieck)**.** *The functor* $\mathbf{Hilb}_{P,r}$ *is representable by a projective scheme* $\mathrm{Hilb}_{P,r}$.

Proof. We sketch the proof, following [16, Theorem 1.9]. Given $Z \subseteq \mathbb{P}^r$ with Hilbert polynomial P, denote the corresponding ideal $I(Z) \subseteq S$. Let

$$O(m) = \binom{r+m}{m}, \quad Q(m) = O(m) - P(m).$$

So $O(m)$ is the dimension of the degree m-piece S_m (as a k-vector space). For large m, $Q(m)$ is the dimension of the mth graded piece $I(Z)_m$. We assert that there exists an m such that for all closed subschemes Z, $I(Z)$ is generated by $I(Z)_m$. So Z is determined by $I(Z)_m$. We can consider $I(Z)_m$ as a vector subspace of S_m. Hence each Z corresponds to a point of the Grassmannian $\mathrm{Gr}_{Q(m)}(O(m))$. The proof finishes by showing that this defines a closed subscheme of $\mathrm{Gr}_{Q(m)}(O(m))$ which represents $\mathbf{Hilb}_{P,r}$. □

Definition 1.2.4. The scheme $\mathrm{Hilb}_{P,r} = \mathrm{Hilb}_P(\mathbb{P}^r)$ is called a *Hilbert scheme*. Denote a point in $\mathrm{Hilb}_P(\mathbb{P}^r)$ corresponding to the subscheme $\mathbb{V}(I) = Z \subset \mathbb{P}^r$ as $[Z]$ or $[I]$.

There are many generalisations, which can be found in [16, Section 1.B].

Definition 1.2.5. Fixing a subscheme $X \subseteq \mathbb{P}^r$, we can define $\mathrm{Hilb}_P(X)$ parametrising subschemes of Z which are closed in $\mathbb{P}^r$ and have Hilbert polynomial P.

Remark 1.2.6. If X is a closed subscheme of $\mathbb{P}^r$, then the above construction defines the Hilbert scheme of closed subschemes of X with fixed Hilbert polynomial.

1.3. **Tangent spaces to Hilbert schemes.** One virtue of the Hilbert scheme is that we can describe the tangent space at a point $[Z]$ as the space of global sections of the *normal sheaf* $\mathcal{N}_{Z/X}$, which is useful for calculations. Recall that the *tangent space* to a scheme X at a closed point x with residue field k, denoted T_xX, is the set of morphisms $\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to X$ which sends the closed point to x. It has the structure of a k-vector space[5].

Example 1.3.1 (Tangent space to $\mathbb{A}^n$)**.** At a closed point $x \in \mathbb{A}^n$, the tangent space is $T_x\mathbb{A}^n \cong k^n$.

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[5]We can construct the tangent *sheaf* or consider the total space as a scheme, but here we will only be concerned with the vector space structure.

Example 1.3.2 (Tangent space to the Grassmannian)**.** Consider the Grassmannian $\mathrm{Gr}_\ell(r)$ of ℓ-dimensional linear subspaces of an r-dimensional vector space V. The tangent space at a point $[W] \in \mathrm{Gr}_\ell(r)$ is isomorphic to $\mathrm{Hom}(W, V/W)$. Indeed, the Grassmannian $\mathrm{Gr}_\ell(r)$ can be covered by open sets of the form

$$U_T = \{W \in \mathrm{Gr}_\ell(r) | W \cap T = \{0\}\},$$

where $T \subseteq \mathbb{A}^n$ is a linear subspace of dimension $r - \ell$. Fix $W \in U_T$. Then any $W' \in U_T$ can be identified with an element $\varphi \in \mathrm{Hom}(W, T)$ by

$$\varphi \mapsto W_\varphi = \{w + \varphi(w) | w \in W\}.$$

Since W is an affine space of dimension ℓ and T is an affine space of dimension $r - \ell$, we can identify $U_T \cong \mathrm{Hom}(W, T) \cong \mathbb{A}^{\ell(r-\ell)}$. Thus, we may identify the tangent space of U_T at $[W]$ with $\mathrm{Hom}(W, T)$. The space T is naturally isomorphic to the quotient V/W and the result follows.

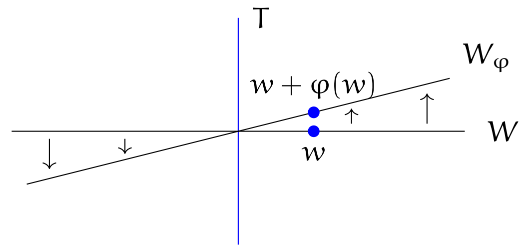


FIGURE 2. Tangent space to Grassmannian considered as elements of $\mathrm{Hom}(W, T)$.

Remark 1.3.3. We can think of V/W as the normal directions to $W \in \mathrm{Gr}_\ell(r)$, and the tangent vector corresponding to $\varphi \in \mathrm{Hom}(W, V/W)$ as specifying a deformation of W into a "nearby" linear subspace by shifting in the normal direction at each point. This idea generalises to Hilbert schemes.

Definition 1.3.4. Let Z be a closed subscheme of scheme X. The *normal sheaf* is

$$\mathcal{N}_{Z/X} = \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}, \mathcal{O}_Z),$$

where $\mathcal{I} = \mathcal{I}_{Z/X}$ is the ideal sheaf of Z in X.

In particular, if $X = \mathrm{Spec}\, A$ is affine and $Z = \mathbb{V}(I)$ for an ideal $I \subseteq A$, then $\mathcal{N}_{Z/X}$ is the sheaf associated to the A-module $\mathrm{Hom}_A(I, A/I)$.

By the universal property, a morphism $\mathrm{Spec}(k[\epsilon]/\epsilon^2) \to \mathrm{Hilb}_{P,r}$ sending the closed point (ϵ) to $[Z]$ corresponds exactly to a flat family

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{i} & \mathbb{P}^r \times \mathrm{Spec}(k[\epsilon]/\epsilon^2) \\ & \searrow^{\varphi} & \downarrow \pi_{\mathrm{Spec}(k[\epsilon]/\epsilon^2)} \\ & & \mathrm{Spec}(k[\epsilon]/\epsilon^2) \end{array}$$

whose fibre over $(\epsilon) \in \mathrm{Spec}(k[\epsilon]/\epsilon^2)$ is Z. More generally, for Z a subscheme of X, a flat family $\mathcal{Z} \subseteq X \times \mathrm{Spec}(k[\epsilon]/\epsilon^2) \to \mathrm{Spec}\, k[\epsilon]/\epsilon^2$ whose fibre over (ϵ) is Z is called a *first order deformation of* Z *in* X. We can think of them as infinitesimal deformations of Z.

Theorem 1.3.5. *For* Z *a closed subscheme of* X*, the space of first order deformations of* Z *in* X *is the space of global sections of* $\mathcal{N}_{Z/X}$.

Proof. We follow [10, Theorem VI-29]. Consider a family $\mathcal{Z}$ over $\operatorname{Spec}(k[\epsilon]/\epsilon^2)$, not necessarily flat, with $\mathcal{Z}_{(\epsilon)} = Z$.

For $U \subseteq X$ an affine open, let $V = Z \cap U$ and $\mathcal{V} = \mathcal{Z} \cap (U \times \operatorname{Spec} k[\epsilon]/\epsilon^2)$. Let A be the coordinate ring $\mathcal{O}_X(U)$ and $I = I(V)$. Then $\mathcal{N}_{Z/X}|_V$ is the sheaf associated to $\operatorname{Hom}_A(I, A/I)$. The coordinate ring on $U \times \operatorname{Spec} k[\epsilon]/\epsilon^2$ is $A \otimes k[\epsilon]/\epsilon^2$. Write elements of this ring as $f + \epsilon g$ with $f, g \in A$. We have

$$I(\mathcal{V}) = (f_1 + \epsilon g_1, \dots, f_k + \epsilon g_k).$$

By hypothesis, this pulls back to $I(V)$ under the inclusion $A \to A \otimes k[\epsilon]/\epsilon^2$, so f_i generate $I(V)$.

We claim that there exists an A-module homomorphism $\varphi : I \to A/I$ which sends f_i to g_i if and only if $\mathcal{V} \to \operatorname{Spec} k[\epsilon]/\epsilon^2$ is flat.

Indeed, the morphism $\mathcal{V} \to \operatorname{Spec} k[\epsilon]/\epsilon^2$ is flat if and only if the coordinate ring

$$B = \mathcal{O}_{\mathcal{Z}}(\mathcal{V}) = (A \otimes k[\epsilon]/\epsilon^2)/I(\mathcal{V}) = (A/I(V)) \otimes k[\epsilon]/\epsilon^2$$

is flat over $k[\epsilon]/\epsilon^2$ (Lemma 1.2.1). Since (ϵ) is the only non-zero ideal of $k[\epsilon]/\epsilon^2$, this is equivalent to $(\epsilon) \otimes B \to B$ being injective. i.e. for $f \in A$,

$$\epsilon f \in I(\mathcal{V}) \implies f \in I(V).$$

Suppose there exists a homomorphism $\varphi : I \to A/I$ with $\varphi(f_i) = g_i$. If $\epsilon f \in I(\mathcal{V})$, then

$$\epsilon f = \sum (a_i + \epsilon b_i)(f_i + \epsilon g_i) = \sum a_i f_i + \epsilon \sum (a_i g_i + b_i f_i),$$

for $a_i, b_i \in A$ and $\sum a_i f_i = 0$. Then $\sum a_i g_i = \varphi(\sum a_i f_i) = 0$ and thus $f = \sum b_i f_i \in I(V)$.

Conversely, if B is flat over $k[\epsilon]/\epsilon^2$, then

$$\sum a_i f_i = 0 \implies \epsilon \sum a_i g_i = \sum a_i (f_i + g_i \epsilon) \in I(\mathcal{V}).$$

So $\sum a_i g_i \in I(V)$ and $\sum a_i f_i \mapsto \sum a_i g_i$ gives a well-defined homomorphism $I \to A/I$, as required.

The result follows from the claim. Indeed, flatness is affine local so $\mathcal{Z}$ is a flat family if and only if it can be covered by flat families $\mathcal{V}$. Given a section of $\mathcal{N}_{Z/X}$, take $\mathcal{Z}$ to be given locally by

$$\{f + \epsilon\varphi(f) | f \in I(V)\},$$

which is a flat family by the claim. Conversely, given flat $\mathcal{Z}$, we obtain φ by applying the construction in the claim affine locally and gluing. ☐

Corollary 1.3.6. *The tangent space of* $\operatorname{Hilb}_P(X)$ *at* $[Z]$ *is given as*

$$T_{[Z]} \operatorname{Hilb}_P(X) = H^0(Z, \mathcal{N}_{Z/X}).$$

Corollary 1.3.7. *Given an ideal* I *of* $S = k[x_0, \dots, x_n]$ *and* $Z = \mathbb{V}(I)$, $T_{[Z]}(\operatorname{Hilb}_P(\mathbb{P}^r)) = \operatorname{Hom}_S(I, S/I)$.

Remark 1.3.8. A more concrete viewpoint is considered in [16, Section 1.C]. In the proof of Theorem 1.2.3, the Hilbert scheme is constructed as a subscheme of a Grassmannian, with the degree m part S_m considered as a k-vector space, and ideal $I(Z)_m$ corresponding to closed subscheme Z considered as a vector subspace of S_m for some large m. Applying Example 1.3.2, a tangent vector in $T_{[Z]} \operatorname{Gr}_{Q(m)}(O(m))$ corresponds to $\varphi \in \operatorname{Hom}_k(I(Z)_m, S_m/I(Z)_m)$. We want to know when such a tangent vector lies in the tangent space of $\operatorname{Hilb}_P(\mathbb{P}^r)$. This motivates the definitions, and such a line of reasoning can be used to derive Corollary 1.3.6 for this case.

1.4. **The Hilbert scheme of points.** We arrive at the main object of interest.

Definition 1.4.1. The *Hilbert scheme of points of* X, denoted $\mathrm{Hilb}^d(X)$, is the Hilbert scheme of subschemes of X with constant Hilbert polynomial $P = d$. i.e. the Hilbert scheme of dimension zero subschemes of length d.

Remark 1.4.2. Suppose X is quasi-projective, so open in some projective scheme $\bar{X}$. Then finite closed subschemes of X are also closed subschemes of $\bar{X}$. $\mathrm{Hilb}^d(\bar{X})$ exists, so $\mathrm{Hilb}^d(X)$ also exists.

For $X = \mathrm{Spec}\, A$ and $Z = \mathbb{V}(I)$, we will often interchange $[Z]$ and $[I]$ for the corresponding point in the Hilbert scheme. The *colength* of an ideal I of a k-algebra A is the dimension of A/I as a k-vector space. We may analogously define the *Hilbert scheme of points* of a ring A, $\mathrm{Hilb}^d(A)$, is the Hilbert scheme of ideals I of colength d. It is clear that $\mathrm{Hilb}^d(\mathbb{A}^n) = \mathrm{Hilb}^d(R)$.

The Hilbert scheme of points has many nice properties inherited from being a Hilbert scheme. It is quasi-projective, universal, and we can describe the tangent space at $[Z] \in \mathrm{Hilb}^d(X)$ in terms of the normal sheaf. For any scheme X, $\mathrm{Hilb}^d(X)$ contains points corresponding to subschemes which are collections of d distinct closed points. So $\mathrm{Hilb}^d(X)$ is naturally related to the *symmetric product*

$$S^d(X) = X^d/S_d,$$

the quotient of the d-th power of X by the action of the symmetric group S_d which permutes the points. More precisely, there is a morphism

$$\varphi : (\mathrm{Hilb}^d(X))_{\mathrm{red}} \to S^d(X)$$

called the *Hilbert–Chow* morphism [11]. This sends a $[Z] \in \mathrm{Hilb}^d(X)$ to the points of its support $\{p_i\}$, with multiplicities a_i given by the length of $\mathcal{O}_{Z,p_i}$. Write such points as $\sum a_i p_i \in S^d(X)$.

Proposition 1.4.3. *If X is connected, then $\mathrm{Hilb}^d(X)$ is connected.*

This is true in general for Hilbert schemes, proven by Hartshorne [17]. Fogarty [11] gives a short proof in the specific case of the Hibert scheme of points using the Hilbert–Chow morphism. Since $S^d(X)$ is connected, it suffices to show that the closed fibres of φ are connected. If $\sum a_i p_i$ is a closed point of $S^d(X)$, with p_i distinct points of X and $\sum a_i = d$, then the set theoretic fibre of $\mathrm{Hilb}^d(X)$ over this point is

$$\prod \mathrm{Hilb}^{a_i}(Z_i),$$

where $Z_i = \mathrm{Spec}\, \mathcal{O}_{X,p_i}/\mathfrak{m}_{p_i}^{a_i}$. Thus we reduce to the local case. The result then follows from the following lemma.

Lemma 1.4.4. [11, Proposition 2.2]. *Let A be a finite dimensional, local k algebra. Then $\mathrm{Hilb}^d(A)$ is connected for all d.*

Proof. Let $r = \dim_k(A)$, $\mathfrak{m}$ the maximal ideal of A. $\mathrm{Hilb}^d(A)$ is a closed subscheme of the Grassmannian $\mathrm{Gr}_d(A) = \mathrm{Gr}_d(r)$, considering A as a k-vector space and ideals as vector subspaces, (cf. proof of Theorem 1.2.3). The units $1 + \mathfrak{m}$ form a unipotent group under multiplication. They act on A via multiplication, giving a map

$$\rho : 1 + \mathfrak{m} \to \mathrm{SL}(d).$$

This in turn defines

$$1 + \mathfrak{m} \to \mathrm{SL}\left(\binom{r}{d}\right) \to \mathrm{PGL}\left(\binom{r}{d} - 1\right),$$

where for the first map we take $\wedge^d \rho$, and the second map is the natural projection. This induces a group action on $\mathrm{Gr}_d(A) \subseteq \mathbb{P}(\wedge^d A)$. In words, we are multiplying the basis vectors of a linear subspace by elements of $1+\mathfrak{m}$. The fixed points of this action are the quotients A/V such that $(1+\mathfrak{m})V = V$, which are precisely the ideals of A. Hence, $\mathrm{Hilb}^n(A)$ is given by the set of fixed points.

In general, if we have a unipotent algebraic group G acting on a closed subscheme Z of $\mathbb{P}^N$ induced by a group homomorphism $G \to \mathrm{PGL}(N)$, then the set of fixed points is connected. Fogarty proves this by induction arguments, which we leave for an interested reader. □

Fogarty shows that for a surface S, $\mathrm{Hilb}^d(S)$ is exceptionally well-behaved.

Theorem 1.4.5 (Fogarty [11])**.** *Suppose S is a non-singular surface over a field k. Then $\mathrm{Hilb}^d(S)$ is a non-singular scheme of dimension $2d$.*

Proof. Fogarty uses that $\mathrm{Hilb}^d(S)$ is connected. The open subset of $\mathrm{Hilb}^d(S)$ corresponding to distinct points has dimension $2d$. Thus $\mathrm{Hilb}^d(S)$ has at least one irreducible component of dimension $2d$ (the smoothable component, see section 2.1). It suffices to show that the tangent space to $\mathrm{Hilb}^d(S)$ has dimension $\leqslant 2d$ for any point.

Let $[Z] \in \mathrm{Hilb}^d(S)$ with corresponding sheaf of ideal $\mathcal{I}$, with $\mathrm{Supp}(Z) = \{p_1, \dots, p_t\}$. We have that

$$T_{[Z]} \mathrm{Hilb}^d(S) \cong \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}, \mathcal{O}_S/\mathcal{I}) \cong \prod_i \mathrm{Hom}_{\mathcal{O}_{S,p_i}}(\mathcal{I}_{p_i}, \mathcal{O}_{Z,p_i}).$$

We thus reduce to the following commutative algebra result, whose proof we omit. □

Lemma 1.4.6. [11, Lemma 2.5]. *Let A be a 2 dimensional regular local ring, I an ideal of A which is primary for the maximal ideal $\mathfrak{m}$. If the length of A/I is equal to d, then the length of $\mathrm{Hom}_A(I, A/I) \leqslant 2d$.*

Remark 1.4.7. Nakajima [25] presents a very short proof of Fogarty's result using homological algebra. The exact sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I}_X = \mathcal{O}_Z \longrightarrow 0$$

gives rise to the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) &\longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z) \longrightarrow \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z) \\ \longrightarrow \mathrm{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z) &\longrightarrow \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_Z) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \\ \longrightarrow \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) &\longrightarrow \mathrm{Ext}^2(\mathcal{O}_X, \mathcal{O}_Z) \longrightarrow \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{O}_Z) \longrightarrow 0. \end{aligned}$$

The Euler characteristic $\sum_i \mathrm{Ext}^i(\mathcal{I}_Z, \mathcal{O}_Z)$ is independent of Z. $\mathrm{Ext}^i(\mathcal{O}_X, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_Z) \cong H^i(X, \mathcal{O}_z(m)) = 0$ for sufficient large m (by Serre vanishing). So we have $\mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{O}_Z) = 0$ and $\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) \cong \mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z)$. By duality,

$$\mathrm{Ext}^2(\mathcal{O}_Z, \mathcal{O}_Z) \cong (\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z \otimes K_X))^\vee \cong (\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z))^\vee \cong (\mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_Z))^\vee.$$

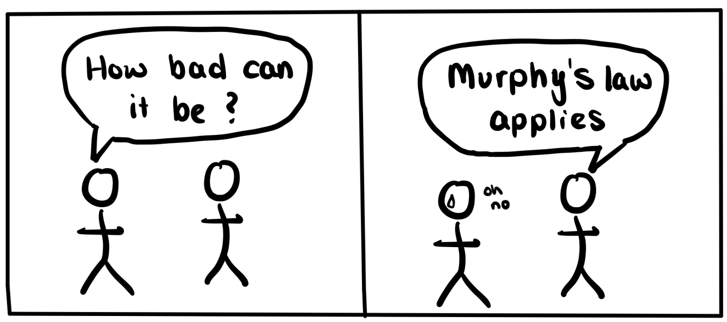
This has dimension d independent of Z. So $\dim \mathrm{Hom}(\mathcal{I}_Z, \mathcal{O}_Z)$ is independent of Z, which implies smoothness.

Another alternative proof for $S = \mathbb{A}^2$ is given in Section 3.3. For surfaces, Fogarty's results have many interesting consequences and the Hilbert scheme of points of a surface is an active area of study. However, this behaviour is an exception rather than a norm. For instance, we have the following result.

Proposition 1.4.8. $\mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^n) = \coprod_{d=1}^{\infty} \mathrm{Hilb}^d(\mathbb{A}^n)$ *is singular for all* $n \geqslant 3$.

We defer the proof to Section 2.1. In fact, we will see that Hilbert schemes of points in general are arbitrarily badly behaved.

2. Pathologies of the Hilbert scheme of points



Law 2.0.1 (Murphy's Law)**.** *Anything that can go wrong, will go wrong.*

In general, Hilbert schemes are rather horrible. Mumford demonstrated a component of a Hilbert scheme which is *everywhere* non-reduced [23] (see also [16, Section 1.D]). Vakil [33] showed that any singularity type, finite type over $\mathbb{Z}$, can be realised on some Hilbert scheme, a result known as *Murphy's Law for Hilbert Schemes*. Vakil also showed that Murphy's law applies to many other well-known moduli spaces. Heuristically, there are two ways in which this arises. Firstly, it is often the case that whenever we wish to parametrise some "nice" geometric objects, in order to obtain a moduli space with nice properties (such as properness), we are forced to throw in some badly behaved objects too. Secondly, moduli spaces can be rather nasty despite parametrising perfectly reasonable objects.

Hilbert schemes of points too inherit this pathological behaviour. We briefly discuss some examples, as well as Jelisiejew's formulation of Murphy's law for the Hilbert scheme of points.

2.1. The shape of $\mathrm{Hilb}^d(X)$.

We start with some simple examples to demonstrate some patterns, before giving examples of more pathological behaviour.

Example 2.1.1. $\mathrm{Hilb}^d(\mathbb{A}^1)$. We are looking for $I \subseteq k[x]$ with finite $\mathrm{colength}(I) = d$. Such I are principal ideals generated by an element of the form

$$f(x) = \prod_{i=0}^{d} (x - a_i),$$

for $a_i \in k$ not necessarily distinct. So $\mathrm{Hilb}^d(\mathbb{A}^1) = S^d(\mathbb{A}^1)$ and has dimension d.

When the a_i are distinct, the resulting ideal corresponds to d distinct points. We can consider "fat" points with multiplicities as the result of some of these points "colliding". In general, we always have a component of the Hilbert scheme of points containing those subschemes which are d distinct points or a result of "collisions". Denote by $\mathrm{Hilb}^d_\circ(X)$ the points in $\mathrm{Hilb}^d(X)$ corresponding to d distinct points and let $\mathrm{Hilb}^d_{sm}(X)$ be its closure.

Proposition 2.1.2 (Jelisiejew [21])**.** *Suppose X is an irreducible, smooth k-scheme. Then $\mathrm{Hilb}^d_{sm}(X)$ is integral for all d. Moreover, if X is a quasi-projective variety of dimension n, then $\mathrm{Hilb}^d_{sm}(X)$ is a quasi-projective variety of dimension dn.*

Remark 2.1.3. We are working over k algebraically closed, but this is not necessary. In general, we require X to be geometrically irreducible.

Proof. See [21, Proposition 4.29]. If X is projective, then $\mathrm{Hilb}^d_{sm}(X)$ is a closed subscheme of a projective scheme and so projective. For X quasi-projective, it is an open set of a projective scheme $\bar{X}$, and $\mathrm{Hilb}^d_{sm}(X)$ is an open set of $\mathrm{Hilb}^d_{sm}(\bar{X})$, hence quasi-projective.

Consider the product of X with itself d times, $X^{\times d}$. Let $X^{d,\circ}$ be the subscheme obtained by removing the (closed) locus where points in the product coincide. The Hilbert–Chow morphism $\mathrm{Hilb}^d(X) \to S^d(X)$ restricts to $\mathrm{Hilb}^d_\circ(X) \to X^{d,\circ}/S_d$. This is an isomorphism, and $X^{\times d}$ is irreducible over k, so it follows that $\mathrm{Hilb}^d_\circ(X)$ is irreducible, and so is $\mathrm{Hilb}^d_{sm}(X)$.

Moreover, $\mathrm{Hilb}^d_\circ(X)$ is smooth. Indeed, considering each distinct point separately, we reduce to the case that $d = 1$. Then $\mathrm{Hilb}^d(X) = X$, which gives the claim. Hence, $\mathrm{Hilb}^d_{sm}(X)$ is reduced. Finally, $\mathrm{Hilb}^d_\circ(X)$ has dimension dn, so $\mathrm{Hilb}^d_{sm}(X)$ has dimension dn too. □

We call $\mathrm{Hilb}^d_{sm}(X)$ the *smoothable component*. We say that a finite closed subscheme Z of a scheme X is *smoothable in* X if there exists an irreducible scheme T and a closed subscheme $\mathcal{Z} \subseteq X \times T$ with a flat family $\mathcal{Z} \to T$ such that:

(1) T has a k-rational point t such that $\mathcal{Z}_t \cong Z$.
(2) For η the generic point of T, $\mathcal{Z}_\eta$ is a smooth scheme over η.

A finite subscheme over k algebraically closed is smooth over k if and only if it is a disjoint union of k-points [21, Lemma 4.22]. So (considering base change) we can think of a smoothable subscheme Z as the limit of d distinct points. Jelisiejew proves the following.

Proposition 2.1.4. [21, Proposition 5.22]. *The smoothable component consists precisely of the smoothable subschemes.*

We return to Proposition 1.4.8.

Proof of Proposition 1.4.8. By the above discussion, it suffices to find a d and an $I \subseteq S$ such that $\dim_k T_{[I]} \mathrm{Hilb}^d(\mathbb{A}^n) = \dim_k \mathrm{Hom}_R(I, R/I) \neq nd$. We take

$$I = (x_1, \dots, x_n)^r,$$

for $r \geqslant 2$. Then $d = \binom{n+r-1}{n}$ and it is shown in [29, Lemma 1.8] that

$$\begin{aligned} \dim_k \mathrm{Hom}_S(I, S/I) &= \binom{n+r-2}{n-1}\binom{n+r-1}{n-1} \\ &= \binom{n+r-2}{n-1} \times \frac{nd}{r}. \end{aligned}$$

For $r \geqslant 2$ and $n > 2$, the right hand side is greater than nd. □

Example 2.1.5. $\mathrm{Hilb}^d(\mathbb{A}^2)$. Again we have points corresponding to d distinct points. However, there are many more ways to have "fat" points. For instance, the ideals

$$I_1 = (x^2, x - y) \quad I_2 = (x, y^2)$$

are both supported at 0 and have colength 2. As in the proof of Theorem 1.4.5, an often useful reduction is to ideals $I \subseteq k[x, y]$ that are supported at a single point. The support of such an ideal $I_i \in \mathrm{Hilb}^d(\mathbb{A}^2)$ is given by the radical $\sqrt{I_i}$. Since we are considering finite subschemes, this is maximal. We have that $\sqrt{I_i} \neq \sqrt{I_j}$ if and only if I_i and I_j are coprime. If $\mathbb{V}(I) = \cup_{i=1}^m \mathbb{V}(I_i)$, then $I = \cap_{i=1}^m I_i$. Moreover,

$$\mathrm{colength}(I) = \sum_{i=1}^{m} \mathrm{colength}(I_i).$$

In the case that I is supported at a single point, S/I is local, with maximal ideal $\sqrt{I}$. Usefully, statements about I_i can be extended to statements about I, for instance:

Proposition 2.1.6. [5, Lemma 4.2]. *Let I be an ideal in $S = k[x_1, \ldots, x_n]$ with decomposition $I = \cap_{i=1}^m I_i \in \mathrm{Hilb}^d(\mathbb{A}^n)$. Then $\mathbb{V}(I)$ is smoothable if each $\mathbb{V}(I_i) \in \mathrm{Hilb}^{d_i}(\mathbb{A}^n)$ is smoothable.*

Restricting our attention to monomial ideals, we can classify using Young diagrams. If I is a monomial ideal, then S/I has a natural basis of monomials $x^a y^b$. The collection of these (a, b) we can put into the diagram. For instance, if $I = (x^3, x^2y, y^2)$, then a monomial basis of S/I is

$$\{1, x, y, x^2, xy\},$$

and the associated Young diagram is given in Figure 3.

y	xy	
1	x	x^2

FIGURE 3. Young diagram for $I = (x^3, x^2y, y^2)$.

Example 2.1.7. $\mathrm{Hilb}^8(\mathbb{A}^4)$. This is an example of a Hilbert scheme of points with a component of dimension smaller than that of the smoothable component, due to Cartwright–Erman–Velasco–Viray [5]. It is not hard to see that $\mathrm{Hilb}^8(\mathbb{A}^4)$ is reducible by demonstrating an ideal at which the tangent space has less than the expected dimension of $4 \times 8 = 32$. The example given in [5, Proposition 5.1] is

$$J = \langle x_1^2, x_1x_2, x_2^2, x_3^2, x_3x_4, x_4^2, x_1x_4 + x_2x_3 \rangle,$$

which has $\mathrm{Hom}(J, R/J) = 32 - 7$.

More generally, they prove the following:

Proposition 2.1.8. [5, Theroem 1.2]. *For $n \geqslant 4$, $\mathrm{Hilb}^8(\mathbb{A}^n)$ has two components: one of dimension $8n$ and another of dimension $8n - 7$.*

We sketch some ideas of the proof. We want to know which [I] are contained in the smoothable component. Proposition 2.1.6 allows us to reduce to the local case. For $(A, \mathfrak{m})$ a local k-algebra, its *Hilbert function* is $h_i = \dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1}$. We will denote this by a tuple $\vec{h}$. Note that (when finite) the dimension of A as a k vector space is $\sum h_i$. We can consider the subscheme $H^d_{\vec{h}} \subseteq \mathrm{Hilb}^d(\mathbb{A}^n)$, which consists of those ideals I supported at the origin, and such that S/I is a local k algebra with Hilbert function $\vec{h}$. These are the *multigraded Hilbert schemes* (see [5, Section 4.2]). We have a closed immersion $H^d_{\vec{h}} \to \mathrm{Hilb}^d(\mathbb{A}^n)$.

Lemma 2.1.9. [5, Theorem 4.23]. *With the exception of local algebras with Hilbert function $(1,4,3)$, every algebra with $d \leqslant 8$ is smoothable.*

The proof is omitted. Cartwright–Erman–Velasco–Viray prove this using a case by case analysis of the possible $\vec{h}$.

Lemma 2.1.10. [5, Proposition 3.3]. *For $e \in \mathbb{Z}_{>0}$, the subscheme $H^{1+n+e}_{(1,n,e)}$ is irreducible.*

Again we omit the proof. The idea is to show that it is isomorphic to $\mathrm{Gr}_{\binom{n+1}{2}-e}(\binom{n+1}{2})$ via functors of points.

In the case of $n = 4, d = 8$, $H^8_{(1,4,3)}$ has dimension 21. We have so far considered ideals supported at the origin, and we can translate to any point of $\mathbb{A}^4$. Heuristically, we obtain an irreducible component of $\mathrm{Hilb}^8(\mathbb{A}^4)$ isomorphic to $H^d_{(1,4,3)} \times \mathbb{A}^4$, which has dimension $21 + 4 = 25 = 32 - 7$. This is proven in [5, Lemma 5.8]. Combining with Lemma 2.1.9, we obtain Proposition 2.1.8 in the case $n = 4$. The case $n \geqslant 4$ is similar.

As a consequence, any $\mathrm{Hilb}^d(\mathbb{A}^n)$ for $d \leqslant 7$ is irreducible. In the reducible case, Cartwright–Erman–Velasco–Viray also describe the intersection. Few other components of smaller dimension are known [21, Section 5.6].

Example 2.1.11. There are also families $Z \hookrightarrow \mathrm{Hilb}^d(X)$ which have dimension larger than dn, where $\dim X = n$.[6] The following construction is adapted from Iarrobino's in [19].

Consider $\mathrm{Hilb}^d(\mathbb{A}^n)$. Let $m \in \mathbb{Z}_{>0}$ and split the degree m monomials of $R = k[x_1, \dots, x_n]$ into two sets

$$\nu_1, \nu_2, \dots, \nu_s \quad \text{and} \quad \mu_1, \mu_2, \dots, \mu_t,$$

with $s = t$, or $s = t + 1$.

There are $\binom{n+m-1}{n-1}$ monomials of degree m, so $s, t \geqslant m^{n-1}/2(n-1)!$. For each $B \in \mathrm{Mat}_{s\times t}$, consider

$$A_B = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_t \end{pmatrix} + B \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_s \end{pmatrix}.$$

Let I_B be the ideal generated by the rows of A_B and $\mathfrak{m}^{m+1}$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Then I_B has colength

$$d = (\#\text{ monomials of degree} \leqslant m \text{ in } R) - t = \binom{m+n}{n} - t \leqslant 2m^n/n!,$$

for $m \geqslant 2n^2$.

[6]This is a different flavour of result to Example 2.1.7. The components that these families are contained in are not known [21, Chapter 1].

For $B \neq B'$, we have $I_B \neq I_{B'}$ and so we have a family of ideals of dimension

$$st \geqslant m^{2n-2}/4((n-1)!)^2,$$

for the given m. Combining these inequalities and setting m sufficiently large, we obtain the dimension of this family as

$$st \geqslant \left(\frac{dn!}{2}\right)^{(2n-2)n} \frac{1}{4((n-1)!)^2} = c(n)d^{2-2/n}.$$

For large d, this is greater than dn, so we have found a family with larger than expected dimension. Iarrobino uses regular local rings to apply this construction to any non-singular projective variety of dimension $n \geqslant 2$.

Example 2.1.12. An explicit example of larger dimension is $\mathrm{Hilb}^{78}(\mathbb{A}^3)$ (see [21, Example 5.44]) found by Iarrabino. The locus irreducible subschemes corresponding to local algebras with Hilbert function (1, 3, 6, 10, 15, 21, 17, 5) has dimension 235, which is one greater than $3 \times 78 = 234$.

To summarise, we have seen that $\mathrm{Hilb}^d(X)$ generally has many components, which includes a smoothable component, and potentially some components of smaller dimension. There may be families too large to fit inside the smoothable component. Moreover, Reeves [28] shows that every component of the Hilbert scheme of points intersects with the smoothable one.

Many questions about the shape of $\mathrm{Hilb}^d(X)$ remain active areas of research, such as which $\mathrm{Hilb}^d(X)$ are irreducible or which ideals are contained in the smoothable component (i.e. which finite algebras over k are smoothable) [21]. We summarise with a picture (see Figure 4).

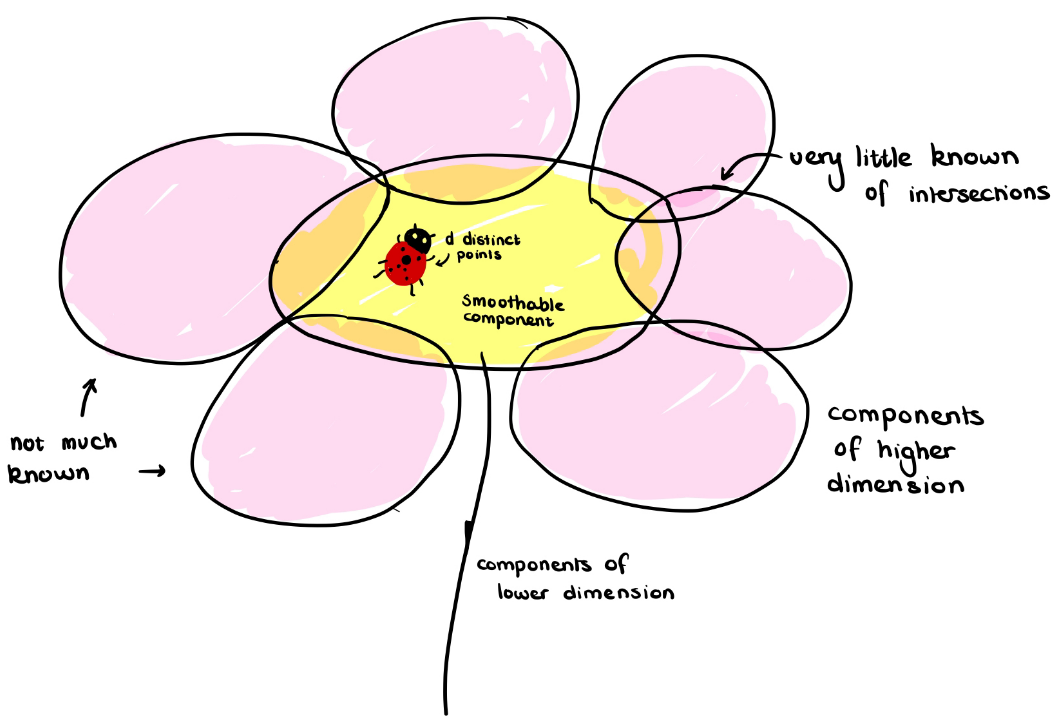


FIGURE 4. The *Bellis Hilbertis*, named by Jelisiejew [21, Page 2].

2.2. **Murphy's Law.** Vakil [33] defines what is meant by *Murphy's law applies* as the following. A *singularity type* is an equivalence class of pointed schemes under the relation generated by: $(X, x) \sim (Y, y)$ if there is a smooth morphism $(X, x) \to (Y, y)$. We say *Murphy's law applies* for a space $\mathcal{M}$ if every singularity type of finite type over $\mathbb{Z}$ appears on this space. Vakil demonstrates many examples of moduli spaces for which Murphy's law applies.

For the Hilbert scheme of points, Jelisiejew [20] defines and proves the following:

Definition 2.2.1. A *retraction* is a morphism of pointed schemes $(X, x) \to (Y, y)$ with a section.

Definition 2.2.2. *Murphy's Law holds up to retraction* for a space $\mathcal{M}$ if, for every singularity type $\mathcal{G}$, there is a representative (Y, y) of $\mathcal{G}$, an open subscheme (X, x) of $\mathcal{M}$, and a retraction $(X, x) \to (Y, y)$.

Theorem 2.2.3 (Jelisiejew [20])**.** *Murphy's Law holds up to retraction for* $\mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$.

Here $\mathrm{Hilb}^{\mathrm{pts}}(X) = \coprod_{d=1}^{\infty} \mathrm{Hilb}^d(X)$. This result extends to $\mathbb{A}^n_{\mathbb{Z}}$ for $n \geqslant 16$.

To understand what this means, we briefly discuss complete local rings. When we want to understand the "local behaviour" of a scheme at a point as in the sense of differential geometry, it is often insufficient to consider the local ring, since the Zariski topology is very coarse. We need to pass to completions. More on completions can be found in [18, Chapter II.9].

Definition 2.2.4. The *complete local ring* of a scheme X at a point x is the completion $\hat{\mathcal{O}}_{X,x}$ of the local ring $\mathcal{O}_{X,x}$, i.e. the inverse limit of $\mathcal{O}_{X,x}/\mathfrak{m}^n_{X,x}$.

For Noetherian schemes, the complete local ring $\hat{\mathcal{O}}_{X,x}$ is regular if and only if $\mathcal{O}_{X,x}$ is regular. We can see how at a regular k point $x \in X$, X "locally[7] looks like affine space" using the following:

Theorem 2.2.5 (Cohen Structure Theorem[8])**.** *If* A *is a complete regular local ring of dimension* n *containing some field, then* $A \cong k[[x_1, \dots, x_n]]$.

Example 2.2.6. Suppose X plane nodal cubic curve given by $y^2 = x^2(x+1)$. Then the complete local ring at the origin is $k[[x, y]]/(xy)$, which corresponds to the fact that near the origin, X looks like two crossing lines.

The following is a corollary of Jelisiejew's result.

Corollary 2.2.7. [20, Corollary 5.1]. *Let* $\mathcal{M} = \mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^{16}_{\mathbb{Z}})$. *For every singularity type* $\mathcal{G}$, *there exists a representative* (Y, y) *of* $\mathcal{G}$ *and a point on* $\mathcal{M}$ *with complete local ring* $\hat{\mathcal{O}}_{Y,y}[[t_1, \dots, t_r]]/I$ *for some* I *such that* $\hat{\mathcal{O}}_{Y,y} \cap I = 0$.

Proof. By Murphy's law up to retraction, we have a retraction $r : (\mathcal{M}, x) \to (Y, y)$ and a section $s : (Y, y) \to (\mathcal{M}, x)$. This induces a homomorphism of local rings $s_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{\mathcal{M},x}$, which is injective since s is a section. $\mathcal{M}$ is finite type of $\mathbb{Z}$, so this implies $\mathcal{O}_{X,x}$ is finitely generated over $\mathcal{O}_{Y,y}$. Now take completions. $\square$

Remark 2.2.8. Heuristically, $X \to Y$ is smooth if X is given by locally adding smooth coordinates to Y (see Figure 5). We may think of two singularity types being equivalent up to retraction if we "add coordinates" that have relations between them. For instance, with $X = \mathbb{V}(xy) \subseteq \mathbb{A}^2$, and any $f : X \to Y$, the projection $X \times Y \to X$ is a retraction, with section given by (id, f). So for k-points $p \in Y$ and $o = \mathbb{V}(x, y) \in X$, the singularity types (X, o) and $(X \times Y, (p, o))$ are equivalent up to retraction. We have that $\mathcal{O}_{X \times Y,(p,o)} \cong \mathcal{O}_{Y,p}[x, y]/(xy)$. We can think of x, y as the new coordinates.

[7] For $k = \mathbb{C}$ this is local in the sense of the analytic topology.

[8] This is a slightly different formulation to usual. See [9, Theorem 7.7].

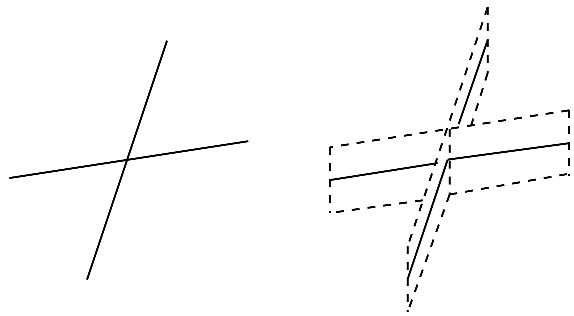


FIGURE 5. Node $X = \mathbb{V}(xy) \subseteq \mathbb{A}^2$ has singularity at origin. The projection $X \times \mathbb{A}^1 \to X$ is smooth, so $(X, 0) \sim (X \times \mathbb{A}^1, (0, 0))$.

As a consequence, we can show the existence of many types of pathological behaviour on $\mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^n_{\mathbb{Z}})$.

Corollary 2.2.9. $\mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^n_{\mathbb{Z}})$ *and* $\mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^n_{\mathbb{C}})$ *are non-reduced for* $n = 16$.

Proof. Apply Theorem 2.2.3 with $[\mathrm{Spec}(\mathbb{Z}[u]/u^2), (u)]$. It suffices to show that if $f : X \to Y$ is a retraction with $x \in X$, $y = f(x)$, then y a non-reduced point if and only if x is non-reduced. This is clear since we have an injective map on stalks $s_y : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. The second result follows by base change. □

We mention some of the ideas of the proof of Theorem 2.2.3. Vakil's method to prove Murphy's law for Hilbert schemes is a "bootstrap" method, which starts with Murphy's law for the *incidence scheme of lines and points*, then successively relates different moduli spaces to each other. Jelisiejew's strategy is an extension of this. There is a $\mathbb{G}_m$ action on $\mathbb{A}^r_{\mathbb{Z}}$ which gives an action on $\mathcal{H} = \mathrm{Hilb}^{\mathrm{pts}}(\mathbb{A}^r_{\mathbb{Z}})$. The *generalised Białynicki–Birula decomposition* gives a scheme $\mathcal{H}^+_{\mathbb{Z}}$ with morphism $\theta : \mathcal{H}^+_{\mathbb{Z}} \to \mathcal{H}$ and a retraction $\pi : \mathcal{H}^+ \to \mathcal{H}^{\mathbb{G}_m}$. Vakil's results can be used to show that Murphy's law holds for $\mathcal{H}^{\mathbb{G}_m}$, but the problem is that θ may not be an open immersion. Jelisiewjew uses *TNT frames* to refine this argument into a proof.

Remark 2.2.10. Jelisiejew's proof is not constructive, although an example of non-reducedness was shown for $\mathrm{Hilb}^{5082}(\mathbb{A}^{14})$ (see [20, Example 5.4]). For an example with smaller d, Szachniewicz [31] uses similar methods to show that $\mathrm{Hilb}^{13}(\mathbb{A}^6)$ is also non-reduced.

3. BOUNDS ON DIMENSION AND MOST SINGULAR POINTS

The singular points of an integral k-scheme X of finite type are those points p where $\dim_k T_p X > \dim X$, so we can view this dimension as some measure of singularity. Murphy's law says that we may have (up to retraction) arbitrary singularities on $\mathrm{Hilb}^{\mathrm{pts}}(X)$. However, we do have a bound on the singularities of $\mathrm{Hilb}^d(X)$ which varies with d, given by Briançon–Iarrobino.

Theorem 3.0.1 (Briançon–Iarrobino [4])**.** *There are non-zero constants* $a(n)$ *and* $b(n)$ *depending only on* n *such that if* $[I]$ *is a point on* $\mathrm{Hilb}^d(\mathbb{P}^n)$ *with maximum dimension tangent space, then*

$$a(n) d^{(2-2/n)} < \dim T(I) < b(n) d^{(2-2/n)}.$$

The lower bound is given by the families discussed in Example 2.1.11. In [4], Briançon–Iarrobino reduce the problem of finding the maximum dimension tangent space to the case of *Borel fixed ideals*. In particular, these are monomial ideals. We discuss this in Section 3.1 and Section 3.2.

In Section 3.4, we discuss Conjecture A, which is that the maximum dimension tangent space is obtained at $\mathfrak{m}^r$. Note that Conjecture A is equivalent to the analogous statement for $\mathfrak{m} = (x_1, \dots, x_n) \subseteq R$ considered as a point of $\mathrm{Hilb}^{\binom{n+r-1}{n}}(\mathbb{A}^n)$. Ramkumar–Sammartano [27] obtain a partial result for $\mathrm{Hilb}^d(\mathbb{A}^3)$ by splitting the tangent space into parts. For I a monomial ideal, $T(I)$ has a natural $\mathbb{Z}^n$-grading, which we describe in Remark 3.3.1.

Definition 3.0.2. [27, Definition 0.1]. A *signature* is a non-constant n-tuple on the two element set $\{p, n\}$. Here p will mean non-negative, n will mean negative. Let $\mathfrak{S}$ be the set of signatures and for each $s \in \mathfrak{S}$, let

$$\mathbb{Z}^n_s = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n : \alpha_i \geqslant 0 \text{ if } s_i = p, \alpha_i < 0 \text{ if } s_i = n)\},$$

$$T_s(I) = \bigoplus_{\alpha \in \mathbb{Z}^n_s} |T(I)|_\alpha \subseteq T(I),$$

where $|T(I)|_\alpha$ is the α graded component of $T(I)$ of degree $\alpha \in \mathbb{Z}^n$.

Theorem 3.0.3 (Ramkumar–Sammartano [27]). *Let $d = \binom{r+2}{3}$ and $[I] \in \mathrm{Hilb}^d(\mathbb{A}^3)$ be a Borel-fixed ideal. Then for $s \in \{ppn, nnp, pnp, npn\}$ and any $r \geqslant 1$,*

$$\dim_k T_s(I) \leqslant \dim_k T_s(\mathfrak{m}^r),$$

and in each case, equality occurs if and only if $I = \mathfrak{m}^r$.

3.1. **Reduction to Borel fixed points.** In the proceeding discussion, we work with $X = \mathbb{P}^n$. The case of $\mathbb{A}^n$ immediately follows. We follow [9, Chapter 15], which has more detailed discussions. Let $G = GL(n+1, k)$. This acts on $S = k[x_0, \dots, x_n]$ via

$$g \cdot \prod_j x_j^{a_j} = \prod_j \big(\sum_{ij} g_{ij} x_i\big)^{a_j}.$$

The *Borel subgroup* $\mathcal{B}$ of G is the subgroup of (invertible) upper triangular matrices.

A *monomial order* is a total ordering on monomials of S satifying: for all $n \neq 1$ a monomial of S and m_1, m_2 monomials with $m_1 > m_2$, have $nm_1 > nm_2 > m_2$. We can extended this notation to terms (i.e. um where $u \in k$ and m a monomial) by saying $um_1 > vm_2$ for $m_1 > m_2$. This is not an ordering, or even a partial ordering, but useful notation.

Example 3.1.1. (1) The *lexicographical ordering* is given by $x_0^{a_0} \dots x_n^{a_n} > x_0^{b_0} \dots x_n^{b_n}$ if one of the following occurs: (i) $\sum a_i > \sum b_i$; or (ii) $\sum a_i = \sum b_i$ and $a_i < b_i$ with the first index i for which $a_i \neq b_i$.
(2) The *reverse lexicographical ordering* is similar to above, but with the last index instead of the first.
(3) A *weight function* is $\lambda : \mathbb{Z}^n \to \mathbb{Z}$ linear. From λ, we obtain a *partial* order, the *weight order associated to* λ, given by $x^a >_\lambda x^b$ if $\lambda(a) > \lambda(b)$.

Remark 3.1.2. While the discussion holds for any monomial ordering, the reverse lexicographical ordering has desirable properties which are useful for applications [9, Chapter 15.7].

Definition 3.1.3. Given a monomial ordering $>$ and $f \in S$, the *initial term* of f, denoted $\mathrm{in}_>(f)$ or $\mathrm{in}(f)$, is the greatest term of f with respect to this ordering. For an ideal I of S, $\mathrm{in}_>(I) = \mathrm{in}(I)$ is the ideal generated by $\mathrm{in}(f)$ for $f \in I$.

Example 3.1.4. Let $n = 3$. For convenience, we denote $x = x_0, y = x_1, z = x_2$, and $\mathbb{P} = \mathrm{Proj}\, k[x, y, z]$. Consider the ideals $I_1 = (x^3, x^2y - y^3, y^2z - xyz)$ and $I_2 = (x - z, y - z)$, and let $I = I_1 I_2$. Then $\mathbb{V}(I)$ is supported at 2 points, $[0 : 0 : 1]$ and $[1 : 1 : 1]$. With respect to the reverse lexicographical ordering,

$$\mathrm{in}(I) = \mathrm{in}(I_1)\,\mathrm{in}(I_2) = (x^4, y^4, xy^3, xy^2z, x^3y, y^3z),$$

which is supported at $[0 : 0 : 1]$.

Remark 3.1.5. We can think of $\mathrm{in}(I)$ as a flat limit. For any I there exists a weight λ such that $\mathrm{in}_{>_\lambda}(I) = \mathrm{in}_>(I)$ (proof omitted, see [9, Proposition 15.16]). For $t \neq 0$, consider the action of the diagonal matrix δ with diagonal entries $(t^{-\lambda_0}, \dots, t^{-\lambda_n})$. For $t \neq 0$, δ is invertible and S/I is isomorphic to $S/\delta(I)$. As $t \to 0$, $\delta(I)$ limits to $\mathrm{in}_{>_\lambda}(I)$. In fact, this describes a flat family of points in $\mathrm{Hilb}^d(\mathbb{P}^n)$ which limits to $[\mathrm{in}(I)]$. Since $\mathrm{in}(I)$ is in the smoothable component, which is connected, this is another proof of the connectedness of $\mathrm{Hilb}^d(\mathbb{A}^n)$. More details can be found in [9, Chapter 15.8].

Remark 3.1.6. We can view this geometrically. Recall that $\mathbb{P}^n$ is a toric variety and there are *1 parameter subgroups* given by

$$\chi : \mathbb{C}^* \to \mathbb{P}^n, t \mapsto [1 : t^{\lambda_1} : \dots : t^{\lambda_n}].$$

These act on $\mathbb{P}^n$ via the torus action. We can consider what happens when t tends to 0 (the flat limit). This sends a point in $\mathbb{P}^n$ to one of the distinguished points, depending on λ_i.

Fix a monomial ordering $>$. Let $\mathcal{U} \subseteq \mathcal{G}$ be the subgroup consisting of upper triangular matrices with 1s down the diagonal.

Theorem 3.1.7 (Generic initial ideal)**.** *Let $I \subseteq S$ be a homogeneous ideal. There exists a Zariski open U which is Borel fixed, meeting $\mathcal{U}$ non-trivially, and a monomial ideal J such that for all $g \in U$, $\mathrm{in}(gI) = J$.*

Definition 3.1.8. With I, J as above, J is called the *generic initial ideal* of I, denoted $\mathrm{Gin}(I)$.

Proof. We follow [9, Theorem 15.18]. Consider the degree ℓ parts I_ℓ of I and S_ℓ of S. Let $f_1, \dots, f_t$ be a basis for I_ℓ. Let $h = (h_{ij})$ be a matrix of indeterminates. Recall that the *symmetric product* of S_ℓ is given by

$$\wedge^t S_\ell = \mathrm{Span}_k \left(\{a_1 \wedge \dots \wedge a_t | a_i \in S_\ell\} / \sim \right),$$

where $a_1 \wedge \dots \wedge a_t \sim \sigma(a_1) \wedge \dots \wedge \sigma(a_t)$ for all $\sigma \in S_t$. Call an element of the form $m_1 \wedge \dots \wedge m_t$, where m_i are monomials of degree ℓ, a *monomial* of $\wedge^t S_\ell$. For $m \neq 0$, we may write this so that $m_1 > \dots > m_t$ and we can order such expressions lexicographically.

Then $h(f_1) \wedge \dots \wedge h(f_t)$ is a linear combination of monomials with coefficients that are polynomials in the h_{ij}. Suppose that $m_1 \wedge \dots \wedge m_t$ is the earliest monomial that appears (with respect to the ordering) with non-zero coefficients. Let the coefficent be $p_\ell(h_{11}, \dots, h_{rr})$, and let

$$U_\ell = \{g = (g_{ij}) \in \mathcal{G} | p_\ell(g_{11}, \dots, g_{rr}) \neq 0\}.$$

By definition, the degree ℓ part of $\mathrm{in}(gI)$ is $(m_1, \dots, m_t)$ if and only if $g \in U_\ell$. Let J_ℓ be the subspace of S_ℓ spanned by $m_1, \dots, m_t$.

Claim that $J = \oplus J_\ell$ is an ideal (i.e. closed under multiplication). It suffices to show that for all ℓ, $S_1 \cdot J_\ell \subseteq J_{\ell+1}$. Since $U_\ell, U_{\ell+1}$ are dense open, there exists a $g \in U_\ell \cap U_{\ell+1}$. Then $in(gI)_\ell = J_\ell$ and $in(gI)_{\ell+1} = J_{\ell+1}$. As $\mathrm{in}(gI)$ is an ideal, the assertion follows.

Finally, we show that $U = \cap_{\ell=1}^{\infty} U_\ell$ is Zariski open and dense in $\mathcal{G}$. In fact, this intersection is finite. Suppose that J is generated by elements of degree $\leqslant e$. Then given $g \in \cap_{\ell=1}^{e} U_\ell$, $in(gI_\ell) = J_\ell$ for all $\ell \leqslant e$. Thus $in(gI) \supseteq J$, and considering the dimensions of the degree ℓ parts (as k vector spaces), we conclude that $in(gI) = J$. Thus $U = \cap_{\ell=1}^{e} U_\ell$ and result follows.

□

Proposition 3.1.9. *For* k *an infinite field, the generic initial ideal is Borel fixed i.e. for all* $g \in \mathcal{B}$,

$$g(\mathrm{Gin}(I)) = \mathrm{Gin}(I).$$

Proof. This is from [9, Theorem 15.20]. Replacing I by gI if necessary, we may assume $\mathrm{in}(I) = \mathrm{Gin}(I)$. The Borel subgroup $\mathcal{B}$ is generated by (i) diagonal matrices δ with diagonal entries δ_i; and (ii) γ_{ij}, where γ_{ij} has 1s down the diagonal, a single 1 in position i, j with $i < j$, and 0 elsewhere. The diagonal matrices fix all monomials, so it suffices to check that

$$\gamma_{i,j}(\mathrm{in}(I_\ell)) = \mathrm{in}(I_\ell).$$

Write $\gamma = \gamma_{i,j}$. Let $f_1, \ldots, f_t$ be a basis for I_ℓ ordered by $>$ and consider $f = f_1 \wedge \ldots \wedge f_t$. Have that $\mathrm{in}(f) = \mathrm{in}(f_1) \wedge \ldots \wedge \mathrm{in}(f_t)$. If $\gamma(\mathrm{in}(I_\ell)) \neq \mathrm{in}(I_\ell)$, then $\gamma\,\mathrm{in}(f) \neq \mathrm{in}(f)$. Note that terms of $\gamma\,\mathrm{in}(f)$ other than $\mathrm{in}(f)$ are strictly greater than $\mathrm{in}(f)$. Let am be a term in $\gamma\,\mathrm{in}(f)$, with $0 \neq a \in k$ and $m \in \wedge^t S_\ell$.

From the proof of Theorem 3.1.7, $\mathrm{in}(I)_\ell = J_\ell$ is spanned by $m_1, \ldots, m_t$ where

$$m_1 \wedge \ldots \wedge m_t = \max\{n_1 \wedge \ldots \wedge n_t | n_1 \wedge \ldots \wedge n_t \in \mathrm{in}(\wedge^t(gI_\ell)), g \in \mathcal{G}\}.$$

This monomial is $\mathrm{in}(f)$, and hence for all g, m should appear in $g(f)$ with coefficient 0. We will show that for some diagonal matrix $\delta \in \mathcal{G}$, m appears with non-zero coefficient in $\gamma\delta f$, which gives a contradiction. To do this, decompose f as follows. For $n = an_1 \wedge \ldots \wedge n_t$, let the *weight* of n be $w = \prod n_i$. Let f_w be the sum of all terms in f having weight w.

Let w_0 be the weight of $\mathrm{in}(f)$. Claim that $\mathrm{in}(f)$ is the unique term with weight w_0 in f. Indeed, if $n_1 \wedge \ldots \wedge n_t$ is any other term, then $\mathrm{in}(f_1) \geqslant n_1, \ldots, \mathrm{in}(f_t) \geqslant n_t$ and so $\prod \mathrm{in}(f_i) \geqslant \prod n_i$ with equality only if $n_1 \wedge \ldots \wedge n_t = \mathrm{in}(f_1) \wedge \ldots \wedge \mathrm{in}(f_t)$.

For a diagonal matrix δ with diagonal entries $(\delta_0, \ldots, \delta_n)$, we have that $\delta(x_i) = \delta_i x_i$, and so $\delta(f) = \sum w(\delta_0, \ldots, \delta_t) f_w$. Hence

$$\begin{aligned} \gamma\delta(f) &= \sum w(\delta_0, \ldots, \delta_t)\gamma f_w \\ &= w_0(\delta_0, \ldots, \delta_t)\gamma\,\mathrm{in}(f) + \sum_{w \neq w_0} w(\delta_0, \ldots, \delta_t)\gamma f_w. \end{aligned}$$

Then writing a_w for the coefficint of m in γf_w, the coefficient of m in $\gamma\delta(f)$ is

$$a_{w_0} w_0(\delta_0, \ldots, \delta_t) + \sum_{w \neq w_0} a_w w_0(\delta_0, \ldots, \delta_t). \tag{1}$$

Have $a_{w_0} = a$, so (1) is a non-zero polynomial in δ_i. Therefore there must exists a δ where it is non-zero, which gives the required contradiction. □

Remark 3.1.10. It is clear from the definition of $\mathrm{Gin}(I)$ that it is a monomial ideal. It also true in general that any ideal fixed by $\mathcal{B}$ is generated by monomials [9, Theorem 15.23].

An ideal I is *strongly stable* if for any monomial $x^\alpha \in I$ with $\alpha_j > 0$, we have $\frac{x_i}{x_j}x^\alpha \in I$ for all $i < j$. In characteristic 0, Borel fixed ideals are exactly strongly stable ideals. We only need one direction, which we prove now.

Proposition 3.1.11. *A Borel-fixed (monomial) ideal is strongly stable.*

Proof. Apply the transformation $x_j \mapsto x_i + x_j$. Since I is a monomial ideal, every monomial in the resulting sum is contained in I. □

Proposition 3.1.12. *Suppose $I \subseteq S$ is a homogeneous ideal. Then I and $\mathrm{Gin}(I)$ have the same colength.*

Proof. In fact, S/I and $S/\mathrm{Gin}(I)$ have the same Hilbert polynomial (see [9, Theorem 15.3] and [9, Theorem 15.26], from which the proof is adapted). Note that g is an invertible k-linear map, so applying g to S, we see that S/I and S/gI have the same dimension. Hence we may assume that $\mathrm{in}(I) = \mathrm{Gin}(I)$. Let B be the set of monomials not in $\mathrm{in}(I)$. Claim that this gives a basis for S/I.

We will use that any subset of monomials has a least element. This follows from the fact S is Noetherian - if we take a set of monomials, the ideal it spans is generated by a finite subset of these monomials. Take the minimal element of this finite set.

Suppose B doesn't span. Let $g \in S/I$ be an element not in the span of B with representative in S having minimal initial element. Any element of I can be written $f = a\,\mathrm{in}(f) + f'$ where f' consists smaller terms and $0 \neq a \in k$. If $g \in S$ contains a term $b\,\mathrm{in}(f)$ with $0 \neq b \in k$, then it can be replaced with $-bf'/a$ with smaller initial element, which contradicts minimality. Thus f must be spanned by B.

For linear independence, suppose that

$$p = \sum u_i m_i = 0 \in I,$$

with $u_i \in k$, $m_i \in B$. Then $\mathrm{in}(p) \in \mathrm{in}(I)$ is one of the m_i, which gives a contradiction.

Therefore $\mathrm{colength}(I) = |B| = \mathrm{colength}(\mathrm{in}(I))$ as required. □

Proposition 3.1.13. $\dim_k T(I) \leqslant \dim_k T(\mathrm{Gin}(I))$.

Proof. Recall that $T(I) = \mathrm{Hom}_S(I, S/I)$. Firstly, we show $\dim_k T(I) = \dim_k T(gI)$ for all $g \in \mathcal{G}$. Indeed, consider a k-linear map $\mathrm{Hom}_S(I, S/I) \to \mathrm{Hom}_S(gI, S/gI)$, which sends an element $\nu \in \mathrm{Hom}_S(I, S/I)$ to $w \in \mathrm{Hom}_S(gI, S/gI)$ defined as

$$w(g(f)) = g\nu(f).$$

This is well defined since $w(gf) \in gI$ if and only if $\nu(f) \in I$, and S linearity is from the fact $g(ff') = g(f)g(f')$. Since g is invertible, the process can be reversed and the claim follows.

So we may assume that $\mathrm{Gin}(I) = \mathrm{in}(I)$. Since $\mathrm{in}(I)$ is a flat limit (see Remark 3.1.5). The result follows by upper-semi continuity of the dimension of the tangent space (see [18, III Theorem 12.8]). □

Corollary 3.1.14. *The maximum dimension of the tangent space at a point of $\mathrm{Hilb}^d(\mathbb{P}^n)$ is obtained at a Borel fixed ideal. In particular, at a monomial ideal.*

3.2. **Young diagrams and bounds on dimension.** Recall in Example 2.1.5, we represented monomial ideals in $\mathrm{Hilb}^d(\mathbb{A}^2)$ by Young diagrams. More generally, we can visualise monomial ideals in $\mathrm{Hilb}^d(\mathbb{A}^n)$ or $\mathrm{Hilb}^d(\mathbb{P}^n)$ for $n \geqslant 2$ by considering *generalised Young diagrams* made from n-dimensional hypercubes. Generalised Young diagrams can be considered as generalised partitions of d. We will often write λ for both the partition and the Young diagram.

Note that if λ is a Young diagram considered as a subset of $\mathbb{Z}^n_{\geqslant 0}$, then $E = \mathbb{Z}^n_{\geqslant 0} \setminus \lambda$ satisfies $E + \mathbb{N}^n = E$.

Definition 3.2.1. The *corners*[9] of E are the smallest subset F of E such that $E = F + \mathbb{Z}^n_{\geqslant 0}$.

For a monomial ideal I, the corners of its Young diagram gives a minimal generating set [4, Section II]. An example is given in Figure 6.

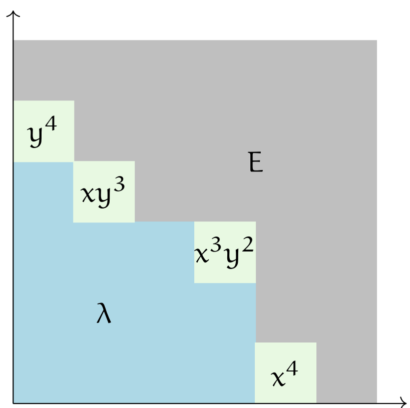


FIGURE 6. Young diagram in blue and corners in green for $I = (x^4, x^3y^2, xy^3, y^4)$.

The *length* of an ideal I in a Noetherian ring is the minimal number of generators. We can bound the dimension of the tangent space at $T_{[I]} \mathrm{Hilb}^d(\mathbb{A})$ in terms of the length of I.

Proposition 3.2.2. *Suppose that* $I \in \mathrm{Hilb}^d(\mathbb{A}^n)$ *has length* $\ell(I)$. *Then* $\dim(T_{[I]} \mathrm{Hilb}^d(\mathbb{A}^n)) \leqslant \ell(I)d$.

Proof. Any morphism of S-modules $I \to S/I$ is determined by the image of the generators of I. Also, S/I has dimension d. The result follows. ☐

This is a very rough bound which may be met, but is certainly not an equality.

Example 3.2.3. Consider $I = (x^2, y^2) \subseteq k[x, y]$. The corresponding Young diagram is in Figure 7.

y	xy
1	x

FIGURE 7. Young diagram for $I = (x^2, y^2)$.

Then $d = \mathrm{colength}(I) = 4$. By Fogarty, we know that the dimension of the tangent space is $8 = 2 \times 4$. We can see this explicitly by defining a morphism $I \to S/I$ which maps x^2, y^2 to any of

[9] These are called *staircases* in [4].

the monomial basis of S/I. There are no relations between x^2 and y^2 so any such choice will give a valid morphism.

Example 3.2.4. Consider $I = (x^2, xy, y^2)$, Young diagram given in Figure 8.

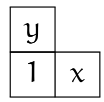


FIGURE 8. Young diagram for $I = (x^2, xy, y^2)$.

We know that the bound of 3×3 is not met by Fogarty. We can see this explicitly. We cannot send x^2 to 0 and xy to 1, since the first implies x^2y must be sent to 0 but the second implies it must be sent to x.

We wish to prove the upper bound of Theorem 3.0.1. Without loss of generality, we may replace $\mathbb{P}^n$ with $\mathbb{A}^n$. The following proposition, combined with the reduction to Borel fixed ideals and Proposition 3.2.2, gives the result.

Proposition 3.2.5. *There are constants $b(n)$ depending only on n, such that if I is a colength d monomial ideal of S that is strongly stable, then the length $\ell(I)$ of I satisfies $\ell(I) \leqslant b(n)d^{1-1/n}$.*

Proof. This is a modified version of [4, Proposition II.1]. Let ν be the largest integer such that for $\mathfrak{m} = (x_1, \dots, x_n)$, $I \subseteq \mathfrak{m}^\nu$ but $I \notin \mathfrak{m}^{\nu+1}$. On the Young diagram, we can see this as the maximal diagram of the from $\mathfrak{m}^\nu$ contained inside λ. Since I is strongly stable, $(\nu, 0, \dots, 0)$ is not in λ. The colength $\binom{\nu+n-1}{n}$ of $\mathfrak{m}^\nu$ is smaller than colength of I, so

$$\nu^n/n! \leqslant ((\nu+1)\dots(\nu+n-1))/n! \leqslant d. \tag{2}$$

We induct on n. For $n = 2$, the number of corners is at most $\nu + 1$, so it follows that

$$\ell(I) \leqslant (2d)^{1/2} + 1 \leqslant (2^{1/2} + 1)d^{1/2},$$

and we may take $b(2) = 2^{1/2} + 1$.

Suppose we have the result in dimension $n - 1$. We "slice" the Young diagram of I. Write $\ell = \ell(I)$. For $i = 0, \dots, \nu - 1$, let

$$\begin{aligned}
\lambda_i &= \lambda \cap (\{i\} \times \mathbb{Z}^{n-1}_{\geqslant 0}),\\
d_i &= \text{size of } \lambda_i,\\
F_i &= \text{staircase of } \lambda_i \in \mathbb{Z}^{n-1}_{\geqslant 0},\\
\ell_i &= \text{cardinality of } F_i.
\end{aligned}$$

Then $d = \sum d_i$ and $\ell \leqslant \sum \ell_i + 1$. By the inductive assumption,

$$\ell \leqslant b(n-1)\Big(\sum d_i^{1-1/(n-1)}\Big) + 1.$$

The map $x \mapsto x^{(1-1/(n-1))}$ is concave, so

$$\begin{aligned}
&\Big(\sum d_i^{1-1/(n-1)}\Big)/\nu \leqslant \Big(\Big(\sum d_i\Big)/\nu\Big)^{1-1/(n-1)}\\
&\implies \sum d_i^{1-1/(n-1)} \leqslant \nu^{1/(n-1)} d^{1-1/(n-1)}.
\end{aligned}$$

Substituting in (2), we obtain

$$\sum d_i^{(1-1/(n-1))} \leqslant (n!)^{1/(n(n-1))} d^{1-1/n},$$

and hence

$$\ell \leqslant b(n-1)(n!)^{1/(n(n-1))} d^{1-1/n} + 1.$$

Thus we may take $b(n) = b(n-1)(n!)^{1/(n(n-1))} + 1$, and this completes the inductive step. □

Remark 3.2.6. Briançon finds explicit values of $b(n)$ in [4], but these are not optimal. A better upper bound of $b(3) = 3.64$ is found for $n = 3$, see [27, Theorem 4.2].

3.3. **Application to surfaces.** The reduction to Borel fixed points, combined with the connection between Hilbert schemes of points and Young diagrams can be used to give an alternative proof for the smoothness of $\mathrm{Hilb}^d(\mathbb{A}^2)$. This proof is given in [14], and also explained in [30, Chapter 18.2].

By our discussion in Section 3.1, we can reduce again to monomial ideals I. Haiman uses coordinates around [I] defined by considering $\mathrm{Hilb}^d(\mathbb{A}^2)$ as a subscheme of the Grassmannian (cf. proof of Theorem 1.2.3). Let V_m be the vector subspace in $k[x, y]$ spanned by the monomials of degree at most m. Set $m > d$. The dimension d subspaces $W \subset V_m$ for which the monomials outside I span V_m/W constitute an affine subvariety U of $\mathrm{Gr}_d(V_m)$ which contains I. Any such W has a unique k-basis consisting of polynomials of the form

$$x^r y^s - \sum_{h,k\in\lambda} c_{hk}^{rs} x^h y^k, \tag{3}$$

for $h, k \in \lambda$, and $r, s \notin \lambda$.[10] The c_{hk}^{rs} form coordinates on the Grassmannian.

To obtain an open affine neighbourhood U of [I] in $\mathrm{Hilb}^d(\mathbb{A}^2)$ we need to consider those W which come from intersecting V_m with an ideal J. Then c_{hk}^{rs} cannot be chosen arbitrarily since J is closed under multiplication by x and y. Multiplying (3) by x and expanding again, we obtain

$$x^{r+1} y^s - \left(\sum_{h+1,k\in\lambda} c_{hk}^{rs} x^{h+1} y^k + \sum_{h+1,k\notin\lambda} c_{hk}^{rs} \sum_{h',k'\in\lambda} c_{h'k'}^{h+1,k} x^{h'} y^{k'} \right). \tag{4}$$

Equating the coefficients with those in

$$x^{r+1} y^s - \sum_{h,k\in\lambda} c_{hk}^{r+1,s} x^h y^k$$

gives relations among the c_{hk}^{rs}. Similarly, we obtain relations by multiplying (3) by y. These relations cut out U as a closed subscheme of $\operatorname{Spec} \mathbb{C}[\{c_{hk}^{rs}\}]$. The point [I] is given by the maximal ideal $\mathfrak{m} = (c_{hk}^{rs})$. Since the cotangent space is given by $\Omega_{\mathrm{Hilb}^d(\mathbb{A}^2),[I]} = \mathfrak{m}/\mathfrak{m}^2$, we can represent a basis of the cotangent space as arrows with tail at $r, s \in \lambda$ and head at box $h, k \notin \lambda$. We can find congruences modulo $\mathfrak{m}^2$ as follows: note that in (4) the double sum lies inside $\mathfrak{m}^2$. Hence moving the tail and head of any arrow one box to the right doesn't change the arrow's class modulo $\mathfrak{m}^2$. Thus, we are free to move the arrow to the right. Similarly, we may move arrows vertically.

[10] Here we write $h, k \in \lambda$ if $x^h y^k \notin I$ (i.e. $h, k \in \lambda$ if the box at coordinates h, k are in the corresponding Young diagram).

Remark 3.3.1. For $n > 2$ we can similarly consider coordinates around [I] for I a monomial ideal given by $c^{r_1,\dots,r_n}_{h_1,\dots,h_n}$, which restrict to elements of $\mathfrak{m}/\mathfrak{m}^2$. By the same analysis, the vectors

$$(h_1 - r_1, \dots, h_n - r_n)$$

are fixed under addition by elements in $\mathfrak{m}^2$, so this defines a grading on $T(I)$.

Ramkumar–Sammartano [27] give a neat presentation. Write $\tilde{I} = \mathbb{Z}^n_{\geqslant 0} \setminus \lambda$. We will say that a *path* between two points $\alpha, \beta \in \mathbb{Z}^n$ is a sequence $\alpha = \gamma^{(0)}, \dots, \gamma^{(m)} = \beta$ such that $\|\gamma^{(i)} - \gamma^{(i-1)}\| = 1$ for all i. A subset $U \subseteq \mathbb{Z}^n$ is *connected* if for any two points in U, there is a path between them. For $V \subseteq \mathbb{Z}^n$, maximal connected subset $U \subseteq V$ is called a *connected component*. A set $U \subseteq \mathbb{Z}^n$ is *bounded* if it is a finite set.

Recall the definition of *signature* in Definition 3.0.2. Now we have

Proposition 3.3.2. *If* $[I] \in \mathrm{Hilb}^d(\mathbb{A}^n)$ *is a monomial point and* $n \geqslant 2$, *then* $T(I) = \oplus_{s \in \mathcal{S}} T_s(I)$.

Proof. The content of the proposition is that we can exclude the constant tuples. No arrow exists with signature $p \dots p$, and any arrow with signature $n \dots n$ can be moved horizontally and vertically until it crosses the axis, so is equivalent to 0. Hence $T_{p\dots p}(I) = T_{n\dots n}(I) = 0$, as required. □

Proposition 3.3.3. [27, Proposition 1.5]. *Let* $\alpha \in \mathbb{Z}^n$ *and* $[I] \in \mathrm{Hilb}^d(\mathbb{A}^n)$. *The set of bounded connected components of* $(\tilde{I} + \alpha) \setminus \tilde{I}$ *corresponds to a basis of* $|T(I)|_\alpha$.

Proof. We give an alternative proof following the discussion above. Consider an arrow in $\mathbb{Z}^n$ with tail in $\tilde{I}$ and head in λ, with direction vector α. This corresponds to an element of $(\tilde{I} + \alpha) \setminus \tilde{I}$ (the position of the head). The unbounded components correspond to arrows whose head crosses the axes, which correspond to zero vectors. If two arrows have heads in the same bounded component of $(\tilde{I} + \alpha) \setminus \tilde{I}$, then they can be moved onto one another via a path with all intermediate arrows having tail in $\tilde{I}$ and head in λ. Hence they are equivalent. Any arrows in different bounded components are not equivalent, since they cannot be translated onto each other in this way. □

Remark 3.3.4. Ramkumar–Sammartano's proof is explicit. For each such bounded component U, can define a map $\varphi_U : I \to S/I$ by setting $\varphi_U(\mathbf{x}^\beta) = \mathbf{x}^{\alpha+\beta}$ if $\alpha + \beta \in U$ and 0 otherwise. It can be shown that this is S linear. Conversely, suppose $\psi \in |T(I)|_\alpha$. If $\alpha + \beta, \alpha + \gamma$ lie in the same connected component $U \subseteq (\tilde{I} + \alpha) \setminus \tilde{I}$, then there exists $c_{\psi,U} \in k$ such that $\psi(\mathbf{x}^\beta) = c_{\psi,U} x^{\alpha+\beta}$ and $\psi(\mathbf{x}^\gamma) = c_{\psi,U} \mathbf{x}^{\alpha+\gamma}$. They then deduce that $\psi = \sum_U c_{\psi,U} \varphi_U$.

We return to the smoothness of $\mathrm{Hilb}^d(S)$. By the discussion of the smoothable component in Section 2.1, it suffices to show that $\dim_k T(I) \leqslant 2d$ for all I.

Proposition 3.3.5. *Let* I *be a monomial ideal of* S. *Then* $\dim_k T_{[I]} \mathrm{Hilb}^d(\mathbb{A}^2) \leqslant 2d$.

Proof. The proof is inspired by [30, Proposition 18.14], modified to use Ramkummar–Sammartano's result. We consider the possible bounded components of $(\tilde{I} + \alpha) \setminus \tilde{I}$ as α varies in $\mathbb{Z}^2$. If α has signature pp or nn, then there are no bounded components. There are two cases remaining.

(1) If α has signature np, then take the top left element of each bounded component U of $(\tilde{I} + \alpha) \setminus \tilde{I}$. So U corresponds to an arrow with head lying just inside λ in column h and tail lying just right of λ in row k.

(2) If α has signature pn, then for each bounded component U, take the bottom right most element of U. This corresponds to an arrow whose head lies just inside row k of λ and whose tail lies just above column h.

Examples of both cases are given in Figure 9. For each type, $(h, k) \in \lambda$ determines the α and the component U uniquely. Thus, there can be at most d basis elements of each type. The dimension bound follows.

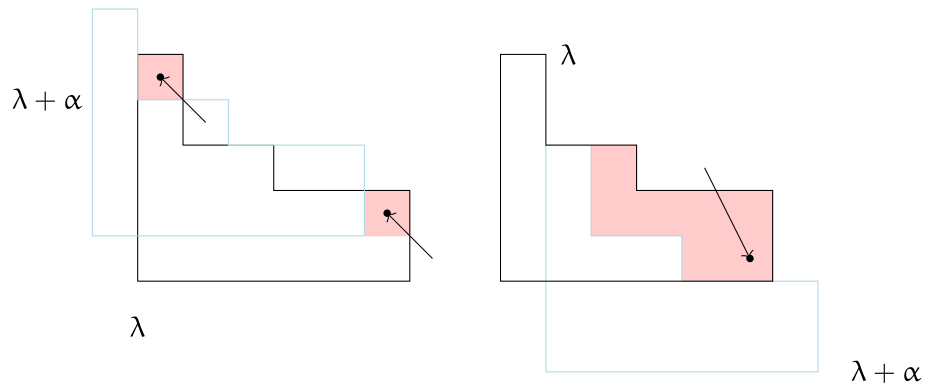


FIGURE 9. Young diagram for $I = (x^6, x^3y^2, xy^3, y^5)$, with bounded components of $(\tilde{I} + \alpha) \setminus \tilde{I}$ shown for $\alpha = (-1, 1)$ and $\alpha = (1, -2)$. We can also consider these elements of $T(I)$ as arrows.

□

3.4. **Results in $\mathbb{P}^3$ and further conjectures.** We now sketch a proof of Theorem 3.0.3, as given by Ramkummar–Sammartano [27].

Lemma 3.4.1. *Let $r \in \mathbb{Z}_{\geqslant 0}$. We have*

$$\dim_k T_{ppn}(\mathfrak{m}^r) = \dim_k T_{pnp}(\mathfrak{m}^r) = \dim_k T_{npp}(\mathfrak{m}^r) = \binom{r+3}{4}$$

$$\dim_k T_{pnn}(\mathfrak{m}^r) = \dim_k T_{npn}(\mathfrak{m}^r) = \dim_k T_{nnp}(\mathfrak{m}^r) = \binom{r+2}{4}.$$

Proof. See [27, Lemma 1.5] and [27, Lemma 3.5]. We give an alternative presentation.

Consider the Young diagram of $I = \mathfrak{m}^r$. We calculate $\dim_k T_{pnp}$. The rest is similar. We must have $|\alpha_i| < r$ for $(\tilde{I} + \alpha) \setminus \tilde{I}$ to be non-empty and the possible cases are $\alpha_1 = 0, \ldots, r-1$. Let $S_x = \{x\} \times \mathbb{R} \times \mathbb{R}$ be a "slice" of the diagram. Then for any $x = 0, \ldots, r-1$, we have that $\lambda \cap S_x$ is a staircase of size $r - x$.

For $x = 0, \ldots, r$ consider

$$((\tilde{I} + \alpha) \setminus \tilde{I}) \cap S_x = ((\tilde{I} + \alpha) \cap S_x) \setminus (\tilde{I} \cap S_x).$$

Then $(\tilde{I} + \alpha) \cap S_x$ is a staircase of size $r - x$ and $((\tilde{I} + \alpha) \cap S_x)$ is a staircase of size $r - x + \alpha_1$ if this is non-negative, and is empty otherwise. If it is nonempty, then we can slide the larger staircase up and left, as in Figure 10.

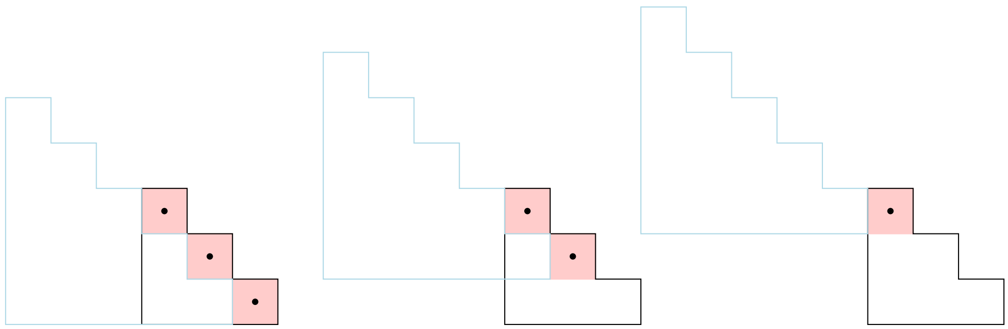


FIGURE 10. The slice $I = \mathfrak{m}^5 \cap S_2$ is a staircase. The bounded components of $(\tilde{I} + \alpha) \setminus \tilde{I}$ is shown for $\alpha = (2, -3, 0), (2, -4, 1), (2, -5, 3)$.

This gives $\binom{r-x+1}{2}$ bounded components. So in total

$$\dim_k T_{pnp} = \sum_{\alpha_1=0}^{r-1} \sum_{x=\alpha_1}^{r-2} \binom{r-x+1}{2} = \sum_{\alpha_1=0}^{r-1} \binom{r+2-\alpha_1}{3} = \binom{r+3}{4}.$$

□

Example 3.4.2. Consider $I = (x, y, z)^4 \subseteq k[x, y, z]$, $\alpha = (-1, 0, 0)$. Then $\tilde{I}$ and $\tilde{I} + \alpha$ are shown in Figure 11, along with the slice at $x = 0$.

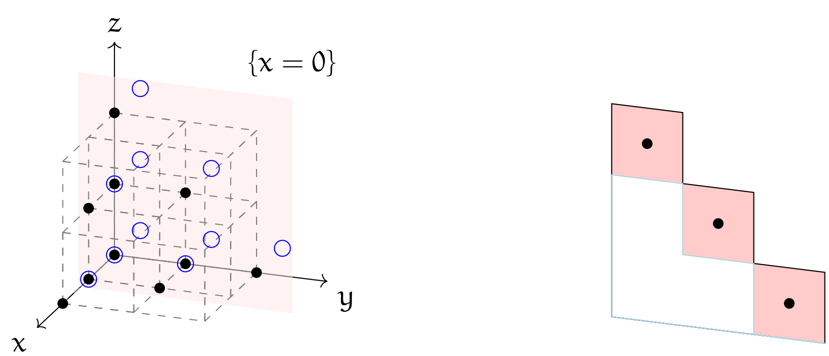


FIGURE 11. $I = \mathfrak{m}^3$, $\alpha = (-1, 0, 0)$. $\tilde{I} + \alpha$ is indicated by the blue circles. Each black point that is not circled is a bounded component of $(\tilde{I} + \alpha) \setminus \tilde{I}$. The slice at $x = 0$ is shown on the right.

Lemma 3.4.3. [27, Theorem 2.4]. *Let* $[I] \in \mathrm{Hilb}^d(\mathbb{A}^3)$ *be a monomial point. We have*

$$\dim_k T_{ppn}(I) = \dim_k T_{nnp}(I) + d$$
$$\dim_k T_{pnp}(I) = \dim_k T_{pnp}(I) + d$$
$$\dim_k T_{npp}(I) = \dim_k T_{pnn}(I) + d.$$

We omit the proof.

Lemma 3.4.4. [27, Proposition 3.1]. *Let* I *be a monomial ideal, and consider the* $k[z]$ *decomposition of* I,

$$I = \bigoplus_{i,j} x^i y^j (z^{b_{i,j}}).$$

For each $\alpha_1, \alpha_2 \geqslant 0$, *we have*

$$\sum_{\alpha_3<0} \dim_k |T(I)|_{(\alpha_1,\alpha_2,\alpha_3)} \leqslant \sum_{i\geqslant\alpha_1, j\geqslant\alpha_2} (b_{i,j} - \max\{b_{i+1,j}, b_{i,j+1}\}). \tag{5}$$

Proof. We present Ramkummar–Sammartano's proof. Fix α_1, α_2 non-negative and define

$$\mathcal{C} = \bigcup_{\alpha_3<0} \mathcal{C}_{\alpha_3} = \bigcup_{\alpha_3<0} \{\text{bounded components of } (\tilde{I} + (\alpha_1,\alpha_2,\alpha_3)) \setminus \tilde{I}\}$$

$$\mathcal{S} = \bigcup_{i\geqslant\alpha_1, j\geqslant\alpha_2} \mathcal{S}_{i,j} = \bigcup_{i\geqslant\alpha_1, j\geqslant\alpha_2} \{(i,j,k) \in \lambda : (i+1,j,k), (i,j+1,k) \in \tilde{I}\}.$$

For each i, j, have that

$$|\mathcal{S}_{i,j}| = b_{i,j} - \max\{b_{i+1,j}, b_{i,j+1}\}$$

Indeed, considering a slice of $\mathbb{Z}^n$ with $z = k$, we may think of an element in the left hand set as an "edge" of the Young diagram. For a fixed i, j, such an edge exists for $z = k$ if $k > b_{i+1,j}, b_{i,j+1}$ (see Figure 12).

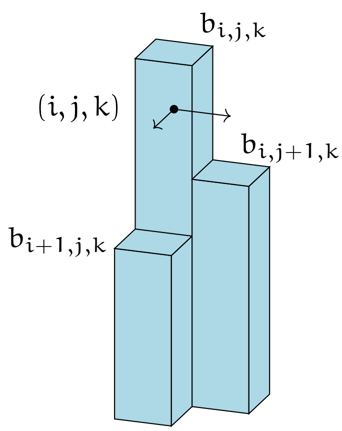


FIGURE 12. A "corner" of λ at (i,j,k) i.e. $(i,j,k) \in \lambda$, $(i+1,j,k), (i,j+1,k) \in \tilde{I}$.

By Proposition 3.3.3, $|\mathcal{C}|$ is equal to the left hand side of (5). So it suffices to show $|\mathcal{C}| \leqslant |\mathcal{S}|$. We define an injective map $\psi : \mathcal{C} \to \mathcal{S}$ as follows: $\psi(U) = (\psi_1(U), \psi_2(U), \psi_3(U))$, where $\psi_3(U)$ is least among the vectors in U with

$$(\psi_1(U)+1, \psi_2(U), \psi_3(U)), (\psi_1(U), \psi_2(U)+1, \psi_3(U)) \notin U$$

This is possible since $|U| < \infty$ and each such bounded $U \subseteq \lambda$ is adjacent to $\tilde{I}$.

Finally, we show ψ is injective. Suppose $U \in \mathcal{C}_{\alpha_3}$, $U' \in \mathcal{C}_{\alpha_3'}$ with $U \neq U'$. If $U \cap U' = \emptyset$, then clearly $\psi(U) \neq \psi(U')$. If $U \cap U' \neq \emptyset$, then we must have $\alpha_3 \neq \alpha_3'$. Without loss of generality, say $\alpha_3 < \alpha_3'$. Then $U' \subset U$ and $\psi(U') + (0,0,\alpha_3 - \alpha_3') \in U$. (See example 3.4.5.)

Hence $\psi_3(U) \leqslant \psi_3(I') + \alpha_3 - \alpha_3' < \psi_3(U')$, so $\psi_3(U) \neq \psi_3(U')$ as required. The result follows. ∎

Example 3.4.5. Consider $I = (x^3, z^3, y^4, x^2y, xy^2, x^2z, y^2z^2)$. Let $\alpha_1 = 3, \alpha_2 = 0$. The slice at $x = 3$ is given in Figure 13, with $\alpha_3 = -2, \alpha_3' = -1$. A translation $(0, 0, -1)$ maps the components $U' \subseteq (\tilde{I} + \alpha) \setminus \tilde{I}$ into $U \subseteq (\tilde{I} + \alpha') \setminus \tilde{I}$.

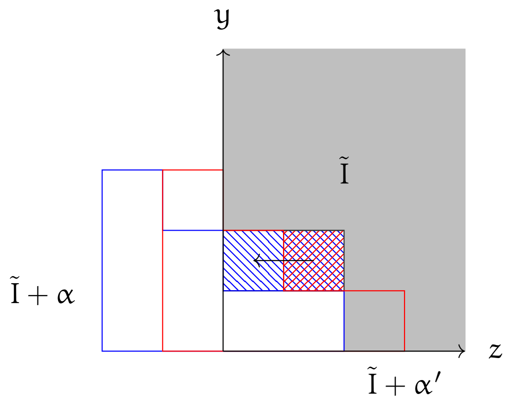


FIGURE 13. A slice at $x = 3$. The blue hatched area U is (a slice of) a bounded component for $(\tilde{I} + \alpha) \setminus \tilde{I}$, the red hatched area U' is (a slice of) a bounded component $(\tilde{I} + \alpha') \setminus \tilde{I}$. After translation $(0, 0, \alpha_3 - \alpha_3')$ U' stays in U.

So far we have only used reduction to monomial ideals, but we will now make use of the reduction to Borel-fixed ideals (Proposition 3.1.9).

Lemma 3.4.6. [27, Lemma 3.2.]. *If* I *is a Borel-fixed (monomial) ideal in* $\mathrm{Hilb}^d \mathbb{A}^3$ *with* $k[z]$ *decomposition*

$$I = \bigoplus x^i y^j (z^{b^z_{i,j}}),$$

then $\max\{b^z_{i+1,j}, b^z_{i,j+1}\} = b^z_{i,j+1}$.

Proof. By Proposition 3.1, I is strongly stable. So $x^i y^{j+1} z^{b^z_{i,j+1}} \in I$ implies $x^{i+1} y^j z^{b^z_{i,j+1}} \in I$, and the result follows. ☐

Proof of Theorem 3.0.3. This is [27, Theorem 3.6]. By Lemma 3.4.3, it suffices to show the inequality for $s = \mathrm{ppn}$ and $s = \mathrm{pnp}$. Consider the $k[z]$, $k[y]$ and $k[y, z]$ decompositions of I

$$I = \bigoplus x^i y^j (z^{b^z_{i,j}}) = \bigoplus x^i z^j (y^{b^y_{i,j}}) = \bigoplus x^i I_i$$

Have $\sum_{j \geqslant 0} b^z_{i,j} = \dim_k(k[y,z]/I_i)$. Applying Lemma 3.4.4 and Lemma 3.4.6, we have

$$\begin{aligned}
\dim_k T_{\mathrm{ppn}}(I) &= \sum_{\substack{\alpha_1, \alpha_2 \geqslant 0 \\ \alpha_3 < 0}} \dim_k |T(I)|_{(\alpha_1, \alpha_2, \alpha_3)} \leqslant \sum_{\alpha_1, \alpha_2 \geqslant 0} \sum_{\substack{i \geqslant \alpha_1 \\ j \geqslant \alpha_2}} (b_{i,j} - \max\{b^z_{i+1,j}, b^z_{i,j+1}\}) \\
&= \sum_{\alpha_1, \alpha_2 \geqslant 0} \sum_{\substack{i \geqslant \alpha_1 \\ j \geqslant \alpha_2}} (b^z_{i,j} - b^z_{i,j+1}) = \sum_{i,j \geqslant 0} (i+1)(j+1)(b^z_{i,j} - b^z_{i,j+1}) \\
&= \sum_{i,j \geqslant 0} (i+1) b^z_{i,j} = \sum_{i \geqslant 0} (i+1) \dim_k \frac{k[y,z]}{I_i} = \sum_{i=1}^{r-1} \sum_{j=i}^{r-1} \dim_k \frac{k[y,z]}{I_j}.
\end{aligned}$$

Claim that for $d \leqslant \dim_k(S/\mathfrak{m}^r)$, for all $0 \leqslant j \leqslant r$, we have

$$\sum_{i=j}^{r-1} \dim_k \frac{k[y,z]}{I_i} \leqslant \sum_{i=j}^{r-1} \dim_k \frac{k[y,z]}{(y,z)^{r-i}},$$

with equality for all i only if $I = \mathfrak{m}^k$. This is [27, Lemma 3.4]. We omit the proof, but it is an induction argument that requires I to be strongly stable. Hence,

$$\dim_k T_{ppn}(I) \leqslant \sum_{i=1}^{r-1} \sum_{j=i}^{r-1} \dim_k \frac{k[y,z]}{(y,z)^{r-i}}. \tag{6}$$

From discussion in Example 3.4.2, $\dim_k T_{ppn}(\mathfrak{m}^r) = \binom{r+3}{4}$. The right hand side of (6) is then

$$\sum_{i=1}^{r-1} \sum_{j=i}^{r-1} \binom{r-j+1}{2} = \sum_{i=0}^{r-1} \binom{r-i+2}{3} = \binom{r+3}{4},$$

from which we obtain the inequality for $s = ppn$. The inequality for $s = nnp$ is shown similarly by using the $k[y]$ decomposition. The final statement is shown by noting that all the inequalities are equalities only if $I = \mathfrak{m}^k$ (we refer the reader to the proof in [27] for details). □

Ramkumar–Sammartano conjecture that the inequalities also hold for $s \in \{npp, pnn\}$. Then Conjecture A would follows from Proposition 3.3.2.

A different approach is taken by Rezaee [29], who conjectured a necessary condition for $T(I)$ to be maximal.

Conjecture 3.4.7. [29, Conjecture B]. *Let $n \geqslant 3$. Suppose that I is a 0-dimensional Borel-fixed ideal in $\mathbb{C}[x_1, \ldots, x_n]$ which is given by*

$$I = (x_1^{m_1}, \ldots, x_n^{m_n}, \textit{all mixed monomial generators}),$$

where $m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n$.

Then, if $\binom{n+r-1}{n} \leqslant \operatorname{colength}(I) \leqslant \binom{n+r}{n}$ and $T(I)$ is maximal, then $m_1 = r$.

Conjecture A follows from this conjecture by applying the following lemma:

Lemma 3.4.8. [29, Lemma 1.7]. *Let $n \geqslant 3$. Suppose that I is a 0-dimensional Borel-fixed ideal in $\mathbb{C}[x_1, \ldots, x_n]$ which is given as in Conjecture 3.4.7. If $\operatorname{colength}(I) = \binom{n+r-1}{n}$, then $I = \mathfrak{m}^r$ is the only ideal of this colength for which $m_1 = r$.*

4. Punctual Hilbert Schemes

We finish with a brief discussion on the punctual Hilbert scheme[11], which we can study with similar techniques.

Definition 4.0.1. The d*th punctual Hilbert scheme*, denoted $P_d(X)$, parametrises length d subschemes supported at a single point of X.

[11] A warning: the Hilbert scheme of points is also often referred to as the punctual Hilbert scheme in the literature. The punctual Hilbert scheme is also referred to as the *local punctual Hilbert scheme* to distinguish the two.

Alternatively, $P_d(X)$ is the reduced fiber of the Hilbert–Chow morphism $\varphi : \mathrm{Hilb}^d(X) \to S^d(X)$ over a multiplicity d cycle, and is also the Hilbert scheme of points of a *local* k-algebra. Many proofs regarding the Hilbert scheme of points or their applications involve the Hilbert–Chow morphism or reducing to the local case (including [4, 5, 11, 15]), which motivates the study of punctual Hilbert scheme. Other authors have studied the punctual Hilbert scheme as an independent object of interest [3, 36].

The tangent space to $P_d(X)$ is comprised of those tangent vectors in $T_{[Z]}\,\mathrm{Hilb}^d(X)$ which are in "vertical" directions relative to φ. More precisely, we have the following general result.

Proposition 4.0.2. *Suppose $f : X \to Y$ is a map of schemes. If $p \in Y$ and q is in the scheme theoretic fibre $f^{-1}(p)$, then*

$$T_q f^{-1}(p) \cong \mathrm{Coker}(df|_q : f^*\Omega_Y|_q \to \Omega_X|_q)^\vee.$$

In particular, to calculate the dimension of $T_q f^{-1}(p)$ it suffices to calculate the dimension of $\mathrm{Coker}(df|_q)$.

So if we consider the map induced by the Hilbert–Chow morphism on cotangent spaces,

$$d\varphi : \Omega_{S^d(\mathbb{A}^2)} \to \Omega_{\mathrm{Hilb}^d(\mathbb{A}^2)},$$

we have that $\dim_k T_{[Z]}(P_d(X)) = \dim_k T_{[Z]}\,\mathrm{Hilb}^d(X) - \dim_k \mathrm{im}(d\varphi|_{[Z]})$. In general, calculating $\dim_k T_{[Z]}(P_d(X))$ in this way requires calculating the dimension of $\dim_k T_{[Z]}\,\mathrm{Hilb}^d(X)$, which we have seen is generally not straightforward, but in the case where X is a smooth surface, we have $\dim_k T_{[Z]}\,\mathrm{Hilb}^d(X) = 2d$. Bejleri–Stapleton use this line of reasoning to prove the following result.

Theorem 4.0.3 (Bejleri–Stapleton [3]). *If I is a monomial ideal such that $[I] = V(I)$ is a length d subscheme supported at the origin, then*

$$\dim(T_{[I]}P_d(S)) = 2d - A,$$

where if λ is the corresponding Young diagram,

$$A = (\textit{maximum of horizontal steps in } \lambda) + (\textit{maximum of vertical steps in } \lambda).$$

Proof. Without loss of generality, we may take $S = \mathbb{A}^2 = \mathrm{Spec}\, k[x, y]$. The map $d\varphi|_{[I]}$ is induced on cotangent spaces by φ^*. The coordinate ring of $S^d(\mathbb{A}^2)$ is

$$A = \wedge^d k[x, y] = k[x_1, \ldots, x_d, y_1, \ldots, y_d]^{S_d},$$

which is generated by polynomials

$$p_{r,s} = \sum_{i=1}^{n} x_i^r y_i^s.$$

We want to calculate $\varphi^*(p_{r,s})$.

The following is from [14, Proposition 2.2]. For any ideal $I \subseteq S$, we can decompose as $I = \cap I_i$, where I_i relatively coprime, $\mathbb{V}(I_i) = p_i = (a_i, b_i)$, and $\mathrm{colength}(I_i) = d_i$. Note that $k[x, y]/I$ is a direct sum of local rings $k[x, y]/I_i$. View x, y as endomorphisms of $\mathbb{C}[x, y]/I$ via multiplication. Let M_x, M_y respectively be the corresponding matrices. Then M_x, M_y commute. In $k[x, y]/I_i$, we have that $(x - a_i)^r = 0$ for some $r \geqslant 1$, so the eigenvalue at p_i of M_x is a_i (with multiplicity d_i). Similarly, the eigenvalue of M_y at p_i is b_i. Hence,

$$\mathrm{Tr}(M_x^r M_y^s) = \sum_i d_i a_i^r b_i^s = p_{r,s}(\varphi(I)).$$

Thus $\varphi^*(p_{r,s}) = \mathrm{Tr}(x^r y^s)$ and the image of φ^* is spanned by $\mathrm{Tr}(x^r y^s) \mod \mathfrak{m}^2$. Recall in Proposition 3.3.5 we considered coordinates c_{hk}^{rs} around $[I] \in \mathrm{Hilb}^d(\mathbb{A}^2)$. Writing $x^r y^s$ as a matrix using this basis, we can compute the trace as

$$\mathrm{Tr}(x^r y^s) = \sum_{h,k \in \lambda} c_{hk}^{r+h,s+k}.$$

Recall in Section 3.3, we considered elements of $\mathfrak{m}/\mathfrak{m}^2$ as arrows. Under this correspondence, $\mathrm{Tr}(x^r y^s)$ can be viewed as a sum of arrows with slope s/r. If s, r are both non-zero, then the arrows are pointing south-west (has signature nn), and hence is equivalent to 0.

When $s = 0$, we have a sum of horizontal arrows of length r. If $r > \max(\triangle h)$, we can move the arrow up and left until the tail crosses the y axis, and it is equivalent to 0. For $1 \leqslant r \leqslant \max(\triangle h)$ this is not possible, so these give non-zero elements. This is demonstrated in Figure 14.

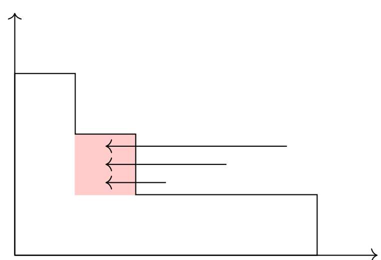


FIGURE 14. Three non-zero elements of $\Omega_{\mathrm{Hilb}^d(\mathbb{A}^2)}$ represented by arrows.

Similarly, we can consider the case $r = 0$, and we obtain a linearly independent generating set for $\mathrm{im}(dh|_{[I]})$ of size $\max(\triangle v) + \max(\triangle h)$. The result follows since $\mathrm{Hilb}^d(\mathbb{A}^2)$ has dimension 2d. □

Example 4.0.4. Consider $I = (x^2, xy, y^3) \in P_4$ and $\alpha = (-1, 0)$. The Young diagram and its translate is given in Figure 15.

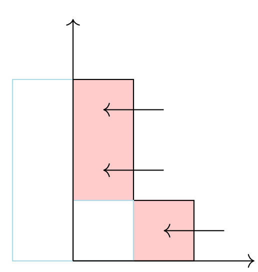


FIGURE 15. The young diagram λ is outlined in black and $(\tilde{I} + \alpha) \setminus \tilde{I}$ is shaded in red for $I = (x^2, xy, y^3)$ and $\alpha = (-1, 0)$.

The two top arrows are equivalent. The arrows correspond to the following morphisms (see Remark 3.3.4):

$$\begin{aligned} \phi_1 : I &\to S/I \\ xy^2 &\mapsto y^2, \\ xy &\mapsto y \end{aligned} \qquad\qquad \begin{aligned} \phi_2 : I &\to S/I \\ x^2 &\mapsto x \end{aligned}$$

All other monomials are sent to 0. By the proof of Theorem 4.0.3, $d(\varphi^*(p_{1,0}))$ corresponds to $\phi_1 + \phi_2$. In light of Section 1.3, this corresponds to the deformation

$$I_\epsilon = (x^2 + \epsilon x, xy + 2\epsilon y, y^3),$$

and for all $0 \neq \epsilon \in k$, $\mathbb{V}(I_\epsilon)$ is supported at two points: the origin and $(-\epsilon, 0)$. We are "splitting" the fat point into two points. Hence this tangent vector is in the cokernel of $d\varphi$ and not contained in the tangent space of P_4.

Corollary 4.0.5. *For any $[I] \in P_d$, we have that $\dim T_{[I]}P_d \leqslant 2d-2$, and equality occurs at a monomial ideal I if and only if $I = \mathfrak{m}^r$ where $\mathfrak{m} = (x, y)$. i.e. $\mathfrak{m}^r$ gives the maximally singular points of P_d.*

Proof. Also see [3, Corollary 11]. The Borel action on $\mathrm{Hilb}^d(\mathbb{A}^2)$ fixes P_d and since P_d is proper, $I \in P_d$ implies $\mathrm{Gin}(I) \in P_d$. So again the maximum dimension tangent space is given at a monomial ideal. Then by Theorem 4.0.3, have

$$\dim T_{[I]}P_d \leqslant 2d-2,$$

with equality when (maximum of horizontal steps in λ) $=$ (maximum of vertical steps in λ) $= 1$ This happens precisely when λ is a staircase, so $I = \mathfrak{m}^r$.

□

Remark 4.0.6. Bejleri–Stapleton show that for $k = \mathbb{C}$, if $\dim T_{[I]}P_d = 2d-2$ for *any* ideal I, then $I = \mathfrak{m}^r$. For this, a more general form of Theorem 4.0.3 is required.

Consider the vector bundle $E \to \mathrm{Hilb}^d(\mathbb{A}^2)$ whose fiber at $[X] \in P_n$ is $H^0(\mathbb{A}^2, T_{\mathbb{A}^2}|_X)$. This is a 2-dimensional vector space. There is a natural injective morphism of sheaves [3]

$$\alpha_d : E \to T_{\mathrm{Hilb}^d(\mathbb{A}^2)}.$$

For a monomial ideal I, we have

$$\mathrm{rank}(\alpha_d|_{[I]}) = (\text{maximum of horizontal steps in } \lambda) + (\text{maximum of vertical steps in } \lambda).$$

This can be proven with similar techniques, showing that derivations act on the Young diagrams by shifting down or right. Details can be found in [3, Section II]. Thus the content of Theorem 4.0.3 is that $\mathrm{corank}(dh|_I) = \mathrm{corank}(\alpha_d|_{[I]})$. Bejleri–Stapleton also prove that for any ideal I,

$$\dim(T_{[I]}P_d) = \mathrm{corank}(d\varphi|_{[I]}) \geqslant \mathrm{corank}(\alpha_d|_{[I]}).$$

The key idea is that there is a holomorphic symplectic form $\omega \in H^0(\mathrm{Hilb}^d(\mathbb{A}^2), \wedge^2\Omega_{\mathrm{Hilb}^d(\mathbb{A}^2)})$, giving an isomorphism $\omega : T_{\mathrm{Hilb}^d(\mathbb{A}^2)} \cong \Omega_{\mathrm{Hilb}^d(\mathbb{A}^2)}$. Bejleri–Stapleton show that $d\varphi$ factors through $\alpha \circ \omega$. 1-parameter degenerations (see Remark 3.1.6) can then used to modify the argument in the proof of Corollary 4.0.5 to give the result.

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