Everything that can go wrong will go wrong

Murphy's Law on Moduli Spaces

Contents

Recommended supplementary material

1. Melody Chan - Moduli Spaces of Curves: Classical and Tropical
2. Ravi Vakil – Murphy's Law in Algebraic Geometry: Badly Behaved Deformation Spaces
3. Robin Hartshorne – Algebraic Geometry
4. Cat videos on instagram

Moduli Spaces

Example: $\mathbb{C}P^n$

$\mathbb{C}P^n$ parametrizes lines in $\mathbb{C}^{n+1}$ going through the origin.

i.e.

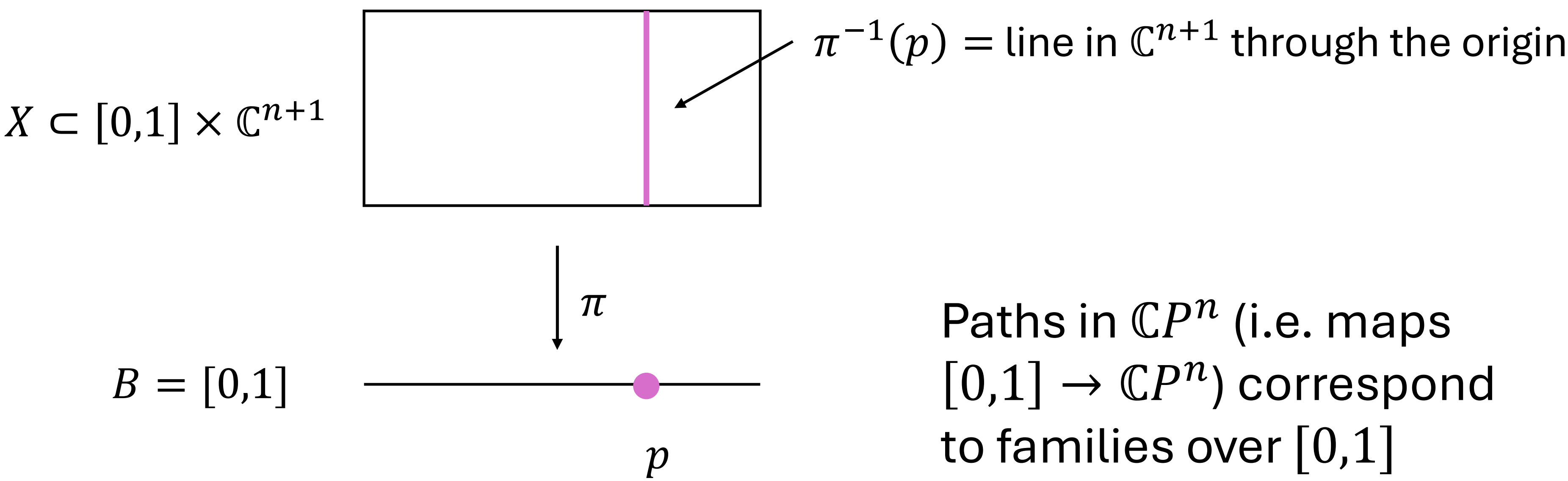
- There is a 1-1 correspondence

$$\{\text{points of } \mathbb{C}P^n\} \leftrightarrow \{\text{lines through the origin}\}$$

- The topology of $\mathbb{C}P^n$ tells you which lines are “close together”

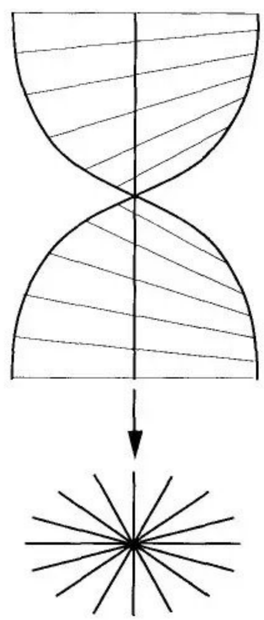
Example: $\mathbb{C}P^n$

Families of lines over $[0,1]$



Paths in $\mathbb{C}P^n$ (i.e. maps $[0,1] \to \mathbb{C}P^n$) correspond to families over $[0,1]$

Example: $\mathbb{C}P^n$



$E = \{(v, l) \in \mathbb{C}^{n+1} \times \mathbb{C}P^n | v \in l\}$

Consider the *tautological bundle* over $\mathbb{C}P^n$.

Given a map $[0,1] \to \mathbb{C}P^n$ we obtain X as the pullback:

$$\begin{array}{ccc} X & \longrightarrow & E \\ \downarrow & & \downarrow \\ B = [0,1] & \longrightarrow & \mathbb{C}P^n \end{array}$$

In general, the *base* B can be an arbitrary space (scheme)

Aside: $E \to \mathbb{C}P^n$ is called a (the) *universal family*

Moduli Spaces

Warning! Speaker is
being very imprecise!

Slogan:

A moduli space is a "nice" space which parametrises the objects you are interested in

A moduli space M may be a variety/scheme/stack/… In particular, you can ask geometric/topological questions about M e.g.

- Dimension of M
- Cohomology of M

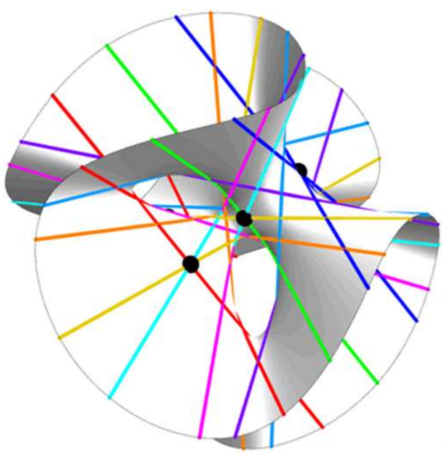
Why suffer through this?

Solutions to many geometric questions in algebraic geometry involve considering an appropriate moduli space.

e.g. 27 lines on a cubic

Ravi Vakil, *Rising Sea,* Chapter 27

David Bai, *Twenty-Seven Lines on a Smooth Cubic Surface*



Incidence correspondence

Z

π ; $\mathbb{P}^{15}$-bundle

Cubic surfaces $\mathbb{P}^{19}$ ← Z → $\mathbb{G}(1,3)$ Lines

Wake an algebraic geometer in the dead of night, whispering: "27". Chances are, he will respond: "lines on a cubic surface".

— R. Donagi and R. Smith, **[DS]** *(on page 27, of course)*

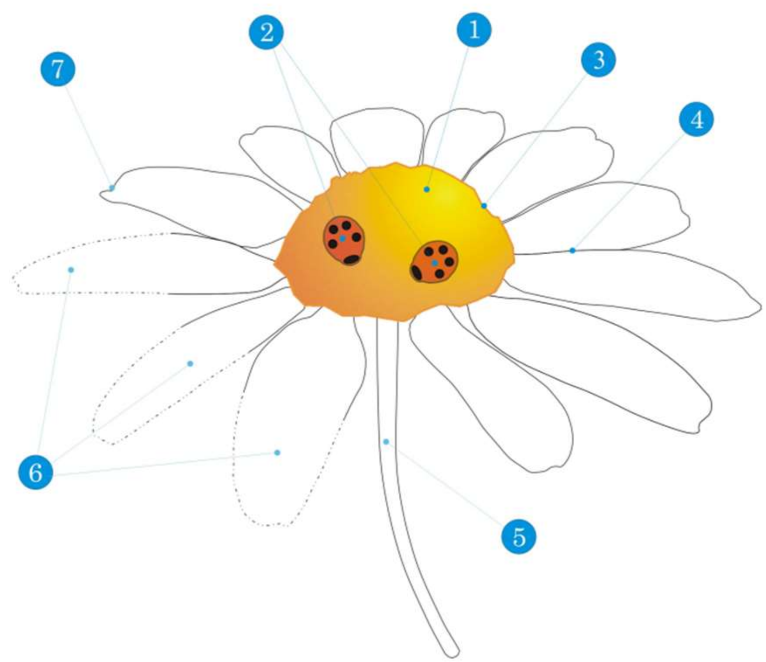
Why suffer through this?

- Moduli spaces are interesting!

[Submitted on 21 May 2022]

Hilbert schemes of points and applications

Joachim Jelisiejew



What is the “shape” of a moduli space
e.g. the Hilbert scheme of points?
What are some salient features?

Example: Hilbert Schemes

Hilbert Schemes are moduli spaces of closed subschemes of projective space (or more generally of projective scheme).

Some remarks:

- Hilbert schemes are schemes! In fact, **projective** (Grothendieck).
- Recall that we can associate to a closed subscheme of projective space a *Hilbert polynomial*. The closed subschemes with the same Hilbert polynomial constitute the connected components of the Hilbert Scheme.

Example: Moduli Space of Curves

Recall that (smooth, complex, projective) curves look like




$\cdots$

$M_{g,n}$ is the moduli space of genus g, n marked (smooth, complex, projective) curves up to isomorphism.

Example: Moduli Space of Maps

Let X be a non-singular projective variety, $\beta \in H_2(X, \mathbb{Z})$.

The moduli space of *stable maps* f from genus g, n marked *nodal* curves to X with $f_*[C] = \beta$ (up to isomorphism) is denoted $\overline{M_{g,n}}(X, \beta)$.

(See next talk :3)

(It will be great fun plz go)

Murphy’s Law

The root of all evil: Hilbert schemes

Law. *There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.*

Often attributed to Mumford.

Mumford's example: everywhere non-reduced component of Hilbert scheme

Moduli of Curves, Chapter 1.D

Consider curves lying on smooth cubic surface S, having class $4H + 2L$ where H is the divisor class of plane section of S (intersect S with a plane) and L a line on S.

The sublocus of the Hilbert scheme defined by these curves can be shown to be:

- Irreducible
- Dense in a component of the Hilbert Scheme

Moreover, at any such curve C, the Hilbert scheme is non-reduced.

Murphy's Law

Consider an equivalence relation on pointed schemes (a scheme with a specified point) generated by: if there is a smooth morphism $X \to Y$ which maps p to q, then $(X, p) \sim (Y, q)$.

Call (X, p) a *singularity* (even if there is no singularity at p!). We call an equivalence class of this relation a *singularity class*.

Definition. *(Vakil) Say* Murphy's Law holds *for a moduli space M if every singularity type of finite type over $\mathbb{Z}$ appears on that moduli space.*

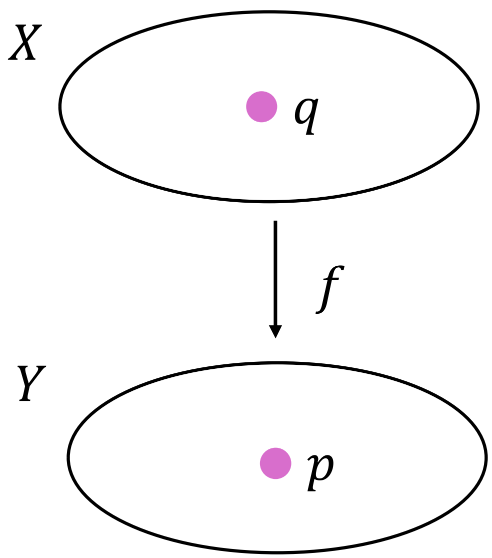
Murphy’s Law

Vakil (‘04) showed that **Murphy’s law holds for many important moduli spaces**, including

- The Hilbert scheme of nonsingular curves in projective space
- The moduli space of maps of smooth curves to projective space
- The versal deformation spaces of smooth surfaces

+ many more examples: see *Murphy’s Law in Algebraic Geometry: Badly-behaved deformation spaces*

Murphy's Law



If f is smooth, then locally $f\colon Spec\ B \to Spec\ A$ where

$$B = A[t_1, \ldots, t_n]/(g_1, \ldots, g_m)$$

where the matrix $\left(\frac{\partial g_i}{\partial x_j}\right)$ has non-vanishing determinant.

Law. *(Vakil) Unless there is some natural reason for a moduli space to be well-behaved, it will be arbitrarily badly behaved*

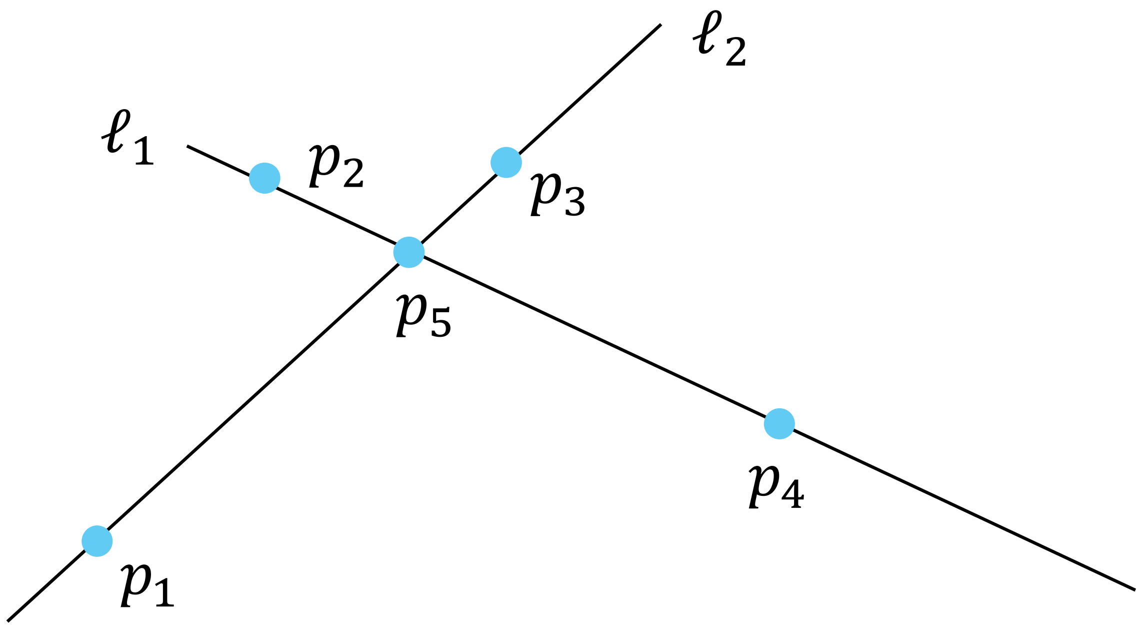
Mnëv's Universality Theorem

Definition. *The* incidence scheme of points and lines in $\mathbb{P}^2$ *is a locally closed subscheme of* $(\mathbb{P}^2)^m \times (\mathbb{P}^{2*})^n = \{p_1, \dots, p_m, \ell_1, \dots, \ell_n\}$ *parametrising* $m \geq 4$ *marked points and* n *marked lines as follows*

- $p_1 = [1:0:0], p_2 = [0:1:0], p_3 = [0:0:1], p_4 = [1:1:1]$
- *Additional information: for each pair* (p_i, ℓ_j) *either* p_i *is required to lie on* ℓ_j *or* p_i *does not lie on* ℓ_j
- *Marked points are distinct, marked lines are distinct*
- *Given two marked lines, there is a marked point required to be on both of them*
- *Each marked line contains at least three marked points.*

Mnëv's Universality Theorem

Consider $m = 5, n = 2$



Mnëv's Universality Theorem

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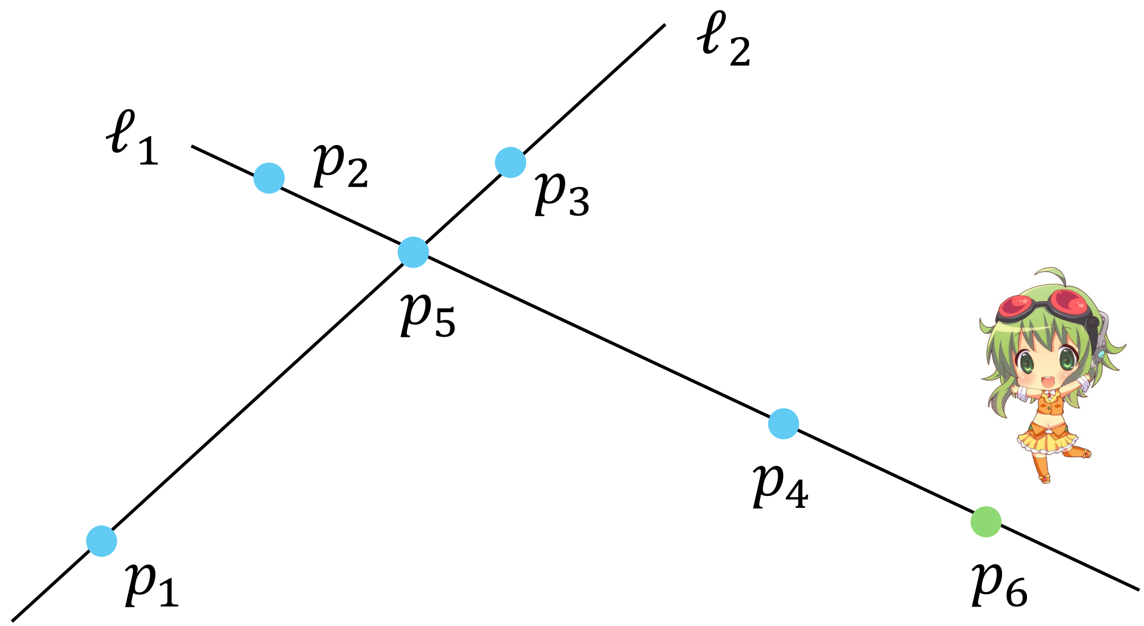

Mnëv’s Universality Theorem

Theorem. (Mnëv) Every singularity type appears on some incidence scheme.

This can be formulated in a more general way.

Vakil's Murphy's Law

We will use Mnëv's theorem to sketch a proof that Murphy's law applies to the moduli space of surfaces with smooth divisors.

Vakil’s Murphy’s Law

Fix a singularity type. By Mnëv’s theorem, there exists an incidence scheme exhibiting this singularity type at a certain configuration $\{p_1, \dots, p_m, \ell_1, \dots, \ell_n\}$

Consider the blow-up of $\mathbb{P}^2$ at the points p_i and let the resulting surface be S.

The lines ℓ_j become smooth curves (proper transforms). Let their union be C.

This induces a morphism from the incidence scheme to the moduli space of surfaces with marked smooth divisors.

Vakil's Murphy's Law

Proposition. This morphism is étale at $\left(\mathbb{P}^2, \{p_i\}, \{\ell_j\}\right) \mapsto (S, C)$

Corollary. The singularity at $\left(\mathbb{P}^2, \{p_i\}, \{\ell_j\}\right)$ has same type as the moduli space of surface with marked smooth divisor at (S, C).

Vakil’s Murphy’s Law

Vakil proceeds to prove that Murphy’s law applies to many other moduli spaces by relating them to each other.

To do this, need to study

- Coverings
- Spaces of deformations

Thank you for listening!

Image references

- Title: CogBlog – A Cognitive Psychology Blog » How to Become More Adaptive to Negative Outcomes in Life (colby.edu)
- Slide 1: Chibi Len Kagamine | Vocaloid Amino (aminoapps.com)
- Slide 3: What algebraic geometry is about | A minimum of blind calculation (wordpress.com)
- Slide 6: "Hatsune Miku Cheer" Sticker for Sale by FribFrog | Redbubble
- Slide 7: 27 Lines on a Cubic Surface | Visual Insight (ams.org)
- Slide 8: Joachim Jelisiejew – Hilbert schemes of points and applications (arxiv.org)
- Slide 10: Kagamine Rin And Len Chibi GIFs | Tenor, Pinterest,
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