Algebraic Curves and Tropical Geometry

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Algebraic varieties are the solution sets of a collection of polynomials[1]. There are two views to take here. On one hand, varieties are spaces, with properties that can be studied with the tools of topology and geometry. On the other hand, polynomials form a ring, and we can use the tools of algebra to study them. Algebraic geometry links the technology developed across many fields, in order to study varieties (which form a large class of interesting spaces), and to generate new results in areas such as algebra and number theory. For instance, the connection between elliptic curves (a well-studied example of an *algebraic curve*) and modular forms was used by Andrew Wiles in his proof of Fermat's Last Theorem. Arithmetic geometry, the study of number theory through algebro-geometric methods, remains hot topic.

Of a different flavour is *enumerative geometry*, which is about counting geometric objects. We have seen shadows of this during the PROMYS programme - we know that a polynomial f over a field has at most $\deg(f)$ solutions. In fact, over the complex numbers[2], f has *exactly* $\deg(f)$ roots, counted with the appropriate multiplicity. A Pokemon evolution of this result is *Bezout's theorem*, which states that two *plane curves* of *degree* d, e respectively meet at de points, counted with the appropriate multiplicity (see Remark 2.3). More generally, intersection theory formalises the idea of "counting intersections" to a diverse range of situations. Another count we are particular interested in is the number of algebraic curves through a fixed number of points, which is of great interest in string theory. This turns out to be quite hard, so many invariants have been concocted which are easier to compute and give the desired count in good situations.

However, the study of varieties is quite hard. A further simplification is obtained via *tropicalisation*, which is a method of associating a combinatorial object to an algebraic variety. The idea is that combinatorics is easier to do than algebraic geometry, and that we can transfer these results to algebraic geometry. This has made great advances in recent years, in simplifying known results [Mik05, NS06], and proving new ones [JR21, Cha21]. In this talk, I will give a brief introduction to algebraic curves and tropical methods.

We will be working over $\mathbb{C}$ unless otherwise specified.

1 Solution sets of polynomials and projective space

A *multi-variate polynomial* is an element of $\mathbb{C}[x_1, \dots, x_n]$. For instance, we have $x^2 + y^2 - 1, xyz + x^3 \in \mathbb{C}[x, y, z]$. The *solution set* or *vanishing set* of a multivariate polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is given by

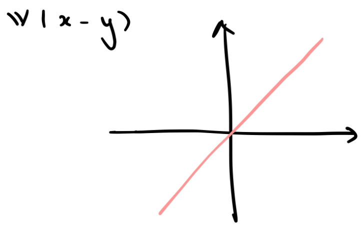
$$\mathbb{V}(f) = \{(x_1, \dots, x_n) \in \mathbb{C}^n | f(x_1, \dots, x_n) = 0\}.$$

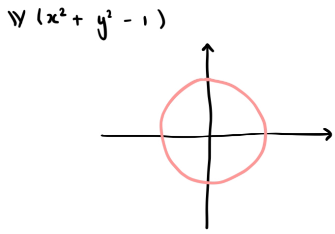
If we have more polynomials, say $f_1, \dots, f_n$, $\mathbb{V}(f_1, \dots, f_n)$ consist of the points which are simultaneous solutions to $f_i(p) = 0$. An *affine variety* is the vanishing set of a set of polynomials[3].

Over $\mathbb{R}$, we can draw some pictures.

Note that these are not representative pictures of these vanishing sets over $\mathbb{C}$. In particular, these vanishing sets will have complex dimension one, so real dimension two, and are some kind of surface.

[1] Classically. We will have no discussion of schemes here.
[2] Or any algebraically closed field.
[3] Without loss of generality, we can set the set to be finite.





Over $\mathbb{C}$, these surfaces are not compact. Here, X is *compact* if every *cover* of X by open sets has a *finite* subcover. We will not go into details about what this means, but for orientation, in $\mathbb{R}^n$ or $\mathbb{C}^n$, a subset is *compact* if it is i) bounded; and ii) every *Cauchy sequence* (roughly, a sequence of points where the points get closer and closer together) has a limit.

Example 1.1. We have some examples of compact and non-compact sets.

Compact	Not Compact
Closed ball	Open ball
A finite set of points in $\mathbb{R}$	$\mathbb{N}$ points at $1/2^n$ on a real line
Torus	$\mathbb{R}^n$

Solution sets to polynomials over $\mathbb{C}$ are almost never compact. This is very sad, for many reasons. For instance, the reason that the statement "two distinct lines meet at a point" isn't true. The counterexample is two distinct parallel lines, which morally meet "at a point at infinity". This point is in some sense "missing", and we want a way of filling it back it.

In general *compactification* refers to a way of taking a space and making it compact. There are many ways to do this, and different methods are good for different reasons. One example is the *one point compactification*, where we add in a "point at infinity". However, for $n > 1$, this compactification of $\mathbb{C}^n$ will be insufficient for our purposes. Intuitively, we do not want skew lines to intersect, but a one point compactification will force them to meet. We will present the construction of projective space, which gives a different compactification of $\mathbb{C}^n$ with many useful properties.

Definition 1.2. *Complex projective space* is (as a set),

$$\mathbb{CP}^n = \{\text{Lines in } \mathbb{C}^{n+1} \text{ passing through the origin}\}.$$

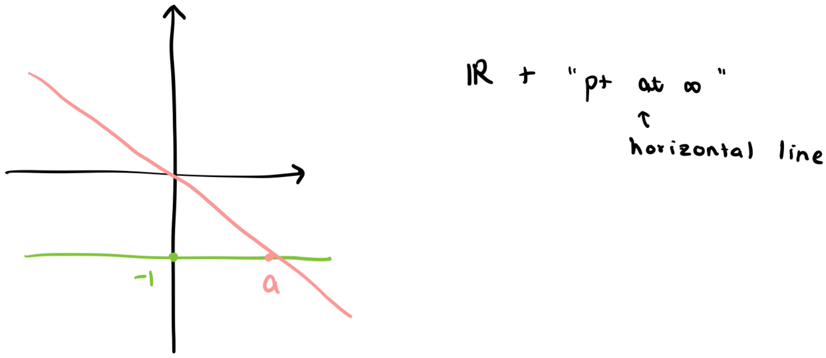
Remark 1.3. To get an intuition for this, lets switch back to $\mathbb{R}$ for a moment. We can, analogously, construct $\mathbb{RP}^n$. A picture of $\mathbb{RP}^1$ is given below. All but one point on $\mathbb{RP}^1$ can be represented by a point on the line $y = -1$. The last point is "at infinity".

Remark 1.4. Projective space has more structure than just a set. In particular, there is a notion of lines being "close together", and we (with some experience) have an intuitive sense that the space of lines should be one dimensional - there is one dimension worth of ways to deform lines through the origin. This is reflected in projective space being some kind of one dimensional space. Projective space is a first instance of a *moduli space*, which parametrises objects of interest and encodes suitable notions of "closeness".

Every such line is determined by a non-zero point in $\mathbb{C}^{n+1}$, but there are many points on the same line. So complex projective space can also be described as

$$\mathbb{CP}^n = \{(x_0 : \ldots : x_n) \in \mathbb{C}^n | x_i \text{ not all zero}\} / \sim .$$

modulo an equivalence relation, where $(x_0 : \ldots : x_n) \sim (\lambda x_0 : \ldots : \lambda x_n)$ for $\lambda \in \mathbb{C}$ non-zero. We denote an equivalence class of this relation as $[x_0 : \ldots : x_n]$. These are called *homogeneous coordinates.*



There is a copy of $\mathbb{C}^n$ contained in $\mathbb{CP}^n$ as follows. Consider U_0, which consists of those elements $[x_0 : \ldots : x_n]$ with $x_0 \neq 0$. We can scale the coordinates so that without loss of generality, $x_0 = 1$. Then

$$U_0 = \{[1 : x_1 : \ldots : x_n] \in \mathbb{CP}^n\} = \mathbb{C}^n.$$

This is the sense in which $\mathbb{CP}^n$ is a compactification of $\mathbb{C}^n$. Note that for $n > 1$ there is more than one point "at infinity". We leave it as an exercise to prove that there is a copy of $\mathbb{CP}^{n-1}$ at infinity, i.e. $\mathbb{CP}^n \setminus \mathbb{C}^n \cong \mathbb{CP}^{n-1}$.

2 Algebraic curves

We return to vanishing sets of polynomials. We want to consider an analogous construction to algebraic varieties in projective space, but we have a problem - the *value* of a polynomial does not make sense! For instance, if we have the polynomials $x^2 - y - 3 \in \mathbb{C}[x, y]$, and we try to substitute in $[2 : 1]$, we obtain a value of 0. But $[2 : 1]$ is equivalent to $[4 : 2]$, and substituting this in, we obtain $11 \neq 0$.

To fix this, we consider *homogeneous* polynomials, where the degree of each term is the same. The value at a point of such a polynomial is still undefined, but the vanishing set is. For instance, if we consider the homogeneous polynomial $F(x, y, z) = x^2 + y^2 - z^2 \in \mathbb{C}[x, y, z]$, we have

$$(\lambda x)^2 + (\lambda y)^2 - (\lambda z)^2 = \lambda^2(x^2 + y^2 - z^2).$$

So (x_0, y_0, z_0) is a solution to $F(x, y, z) = 0$ if and only if $(\lambda x_0, \lambda y_0, \lambda z_0)$ is a solution for all $\lambda \neq 0$. Hence the vanishing set

$$\mathbb{V}(F) = \{[x_0 : \ldots : x_n] \in \mathbb{CP}^n | F(x_0, \ldots, x_n) = 0\}$$

is well-defined. More generally, we obtain $\mathbb{V}(F_1, \ldots, F_n)$, and such spaces are called *projective varieties.*

We have a simple way of turning a homogeneous polynomial in $F \in \mathbb{C}[x_0, \ldots, x_n]$ into a polynomial in $f \in \mathbb{C}[x_1, \ldots, x_n]$, namely setting $x_0 = 1$. In general, we can chose to set any $x_i = 1$. For example, if we set $z = 1$ for the the polynomial $F(x, y, z)$ above, it becomes $x^2 + y^2 - 1$. We call this process *dehomogenisation*. There is a reverse process of *homogenisation*, which we leave to the reader to think about.

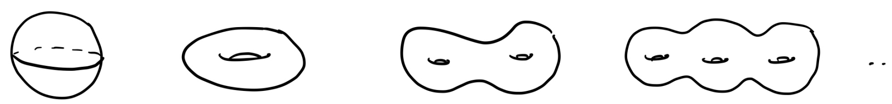
Suppose we have a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ and corresponding homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$. Then a compactification of $X = \mathbb{V}(f)$ is given by $\bar{X} = \mathbb{V}(F)$. This is called the *projective closure* of X.

Remark 2.1. In general, the projective closure of an affine variety $\mathbb{V}(f_1, \ldots, f_n)$ is given by some projective variety, but it is not true that the projective closure is $\mathbb{V}(F_1, \ldots, F_n)$, where F_i is the homogenisation of f_i. This is clunky for good reason - the correct way to think about this is with ideals, but let's not go there now.

Consider a projective variety X. If it is *smooth* (a term we will not define, but heuristically it has no "bad points" such as cusps or nodes), and has (complex) dimension 1, then X is a *algebraic curve* or *Riemann surface*. Algebraic curves are amazingly rich to study. Firstly, we may ask the following:

Question 2.2. *Can we classify algebraic curves?*

Firstly, algebraic curves are compact orientable surfaces (real dimension two) without boundary, which is classified by the *genus*, roughly the number of "holes".



We have the following facts:

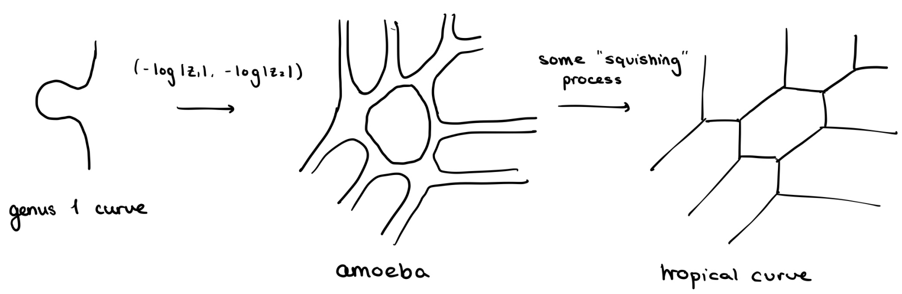
1. An algebraic curve with genus 0 is isomorphic[4] to $\mathbb{CP}^1$.
2. There are infinitely many algebraic curves which have genus 1 that are pairwise non-isomorphic. An *elliptic curve* is a genus 1 algebraic curve along with a chosen point.

Another basic invariant of an algebraic curve is its *degree*, which is the generic number of intersections with a hyperplane.

Remark 2.3. Bézout's theorem states that a degree d plane curve and a degree e plane curve meets at de points, counted with the appropriate multiplicity.

3 A view towards tropical geometry

Algebraic curves are hard to study. The idea of tropical geometry is to reduce algebraic curves to a combinatorial object, so that they are easier to study. To demonstrate some ideas, consider an algebraic curve $X \subseteq \mathbb{CP}^2$ and intersect it with $(\mathbb{C}^*)^2 \subseteq \mathbb{CP}^2$ to obtain X'. We can apply the map $X' \to \mathbb{R}^2$, given by $(z_1, z_2) \mapsto (-\log|z_1|, -\log|z_2|)$. This produces a subset of the real plane called an *amoeba*. A picture is given in



An amoeba has some graph-like structure concentrated near the origin, and some "tendrils" going off to infinity in particular directions. By some suitable limiting process, we can extract a union of intervals

[4] Note that we have not defined what an isomorphism is. This is a sketchy talk for a reason. For details, the writer recommends Vakil or Hartshorne, with particular fondness for Vakil, but suggests taking some courses in algebra and geometry first.

in the plane, which is called an *tropical curve*. A tropical curve has the structure of a graph, with the added information of edge lengths, and some number of infinite *legs*. The process of producing a tropical curve from an algebraic one is called *tropicalisation*.

Tropicalisation is a dramatic operation that kills a lot of information about the algebraic curve. However, it turns out that we retain enough information for this to be useful. For instance, the genus of the algebraic curve is the *genus* of the tropical curve[5]. The degree of the curve is given by the number of legs going off in the directions $(-1,0),(0,-1)$, or $(1,1)$, which is called the *degree* of the tropical curve.

Starting with some question about algebraic curves, tropical methods typical involve the following:

1. Solve an analogous question with tropical curves.
2. Prove a "lifting theorem" which relates the tropical answer to the algebraic answer.

There are many cases where finding and solving the analogous tropical question is much easier, and this method has allowed advances on important questions involving algebraic curves. However, there are two difficulties. Firstly, it is not always clear what the right tropical construction is, and the analogous tropical question can be surprising. For instance, divisor theory on algebraic curves has an analogy with *chip firing configurations* [CDPR12], while it is not clear at all what the right analogy for a line bundle should be on a tropical curve. Secondly, lifting theorems are sometimes highly non-trivial, and while much progress has been made in a number of contexts [CFPU15, Spe14, Ran17], the general picture is still very much obscure.

There is much to talk about here, and this talk is finite, so we will finish by indicating the ideas of one example.

Question 3.1. *Given n fixed points in $\mathbb{P}^2$, how many degree d, genus g curves pass through them?*

We should immediately ask, does this question make sense? In particular, is the number finite? Does it matter what points we choose? These are all important, but we will not discuss them.

Example 3.2. A degree 1 curve is a line. There is 1 line passing through two points, infinitely many lines through one point, and no lines through three general points.

The moral is that we should choose n, d, g wisely. We take $n = 3d + g - 1$. We define $N_{d,g}$ to be the number of degree d, genus g through n generically chosen points in $\mathbb{P}^2$. It turns out that this is well-defined[6]. We write $N_d = N_{d,0}$. We have the following result.

Theorem 3.3 (Kontsevich)**.** *The numbers N_d satisfies the following recursion relation. $N_1 = 1$ and*

$$N_d = \sum_{\substack{d_1+d_2=0\\ d_1,d_2>0}} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^4 d_2 \binom{3d-4}{3d_1-1} \right) N_{d_1} N_{d_2}.$$

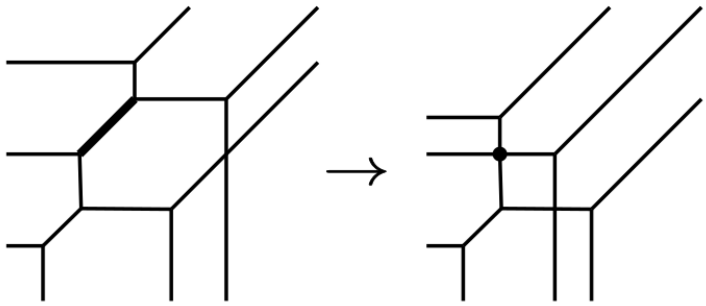
The standard proof requires some amount of technology.

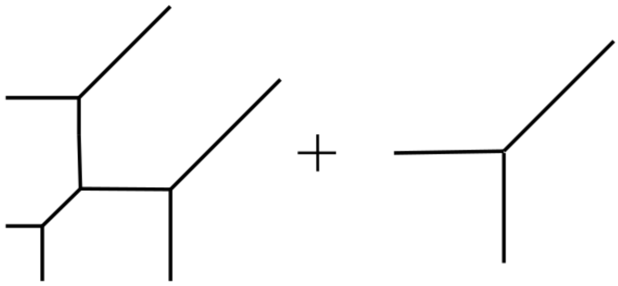
Gathmann-Markwig [GM08] give an alternative proof of this result using tropical curves. For plane tropical curves, the lifting was proven by Mikhalkin[7] [Mik05]. For the analogous tropical question, the rough idea is that a tropical curve of degree d with contracted edges can be "split" into two tropical curves of degree d_1, d_2, with $d_1 + d_2 = d$.

[5] For an appropriate definition of genus.

[6] We may ask this question for a general variety X, instead of just $\mathbb{CP}^2$. It turns out that these questions become significantly harder. A substitute invariant is the Gromov-Witten invariant, which can often actually be computed, and is equal to the count we want in good cases. However, the GW invariant is longer enumerative - it can be a fraction, or even negative, so is clearly no longer counting geometric objects.

[7] This paper was the first use of tropical geometry for algebraic curves, and fundamental to the field. It is surprisingly readable and good for intuition, although the amoeba construction has since succeeded by more readily generalisable constructions.





The proof concludes by a counting argument. As a warning, we are throwing some stuff under the rug and being very vague. For the very enthusiastic, the author recommends Gathmann–Markwig's original paper.

Consider the moduli space (see Remark 1.4) $M_{d,n}$ of n-marked, degree d tropical curves, and let $M_d = M_{d,0}$. We have some natural maps

1. The *forgetful map* $\mathrm{ft} : M_{d,n} \to M_{d,m}$ where $C \in M_{d,n}$ is mapped to $\mathrm{ft}(C) \in M_{d,m}$, where we "forget" that the points $x_{n+1}, \ldots, x_m$ on C exists.
2. The *evaluation maps* ev_i: $M_{d,n} \to \mathbb{R}^2$, where for each tropical curve $C \in M_{d,n}$, we record where x_i is.

Definition 3.4. For $d \geq 2$ and $n = 3d$, define

$$\pi : \mathrm{ev}_1^1 \times \mathrm{ev}_2^2 \times \mathrm{ev}_3 \times \ldots \times \mathrm{ev}_n \times \mathrm{ft}_4 : M_{d,n} \to \mathbb{R}^{2n-2} \times M_4.$$

Here ev_i^j is the jth coordiante of ev_i. We calculate the degree of π at two different points, P and P' on M_4 (roughly, how many preimages each points have, with multiplicity), and they should be the same. The contributions are as follows:

(P) We have (i) The curve is irreducible, or (ii) the curve reduces into two components C_1, C_2, with x_1, x_2 on C_1, and x_3, x_4 are on C_2.

(P') x_1, x_3 are on one component, x_2, x_4 are on the other.

For (P), the irreducible contribution gives N_d. For the reducible contributions, we require $3d_1 - 1$ of the other points $x_5, \ldots, x_n$ to be on C_1 and the other points are on C_2. This is a dimension condition - note we want a finite count! There are $\binom{3d-4}{3d_1-1}$ ways of choosing which points go on which component. There are N_{d_1} choices for C_1, N_{d_2} choices for C_2. For the choice of x_1 and x_2, by the definition of π they must be on fixed lines. By a tropical version of Bézout's theorem (see Remark 2.3), there are d_1 choices for each. Also by Bézout's theorem (see Remark 2.3), C_1 and C_2 intersect at $d_1 d_2$ points. Hence there are $d_1 d_2$ choices for the gluing of C_1 and C_2.

A similar calculation gives the other part of the sum, and equating the two gives the result.

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